HARMONICS,

OR

THE PHILOSOPHY OF MUSICAL SOUNDS.

BY

ROBERT SMITH, D. D, F. R. S,
And Maſter of Trinity College
In the Univerſity of Cambridge.

O Decus Phœbi ————
———— *ô laborum*
Dulce lenimen.

THE SECOND EDITION,
Much improved and augmented.

LONDON:
Printed for T. and J. MERRILL Bookſellers in Cambridge;
Sold by B. DOD, J. WHISTON and B. WHITE,
J. NOURSE, and M. COOPER in London;
J. FLETCHER, and D. PRINCE in Oxford,
MDCCLIX.

Scimus musicen, mathesin, atque adeo veram physicam nostris moribus NON *abesse à Principis persona: Quæ quidem omnia apud Græcos non laude solùm, sed honore et gloria digna ducebantur.*

Epaminondas, Imperator ille insignis, ne dicam summus vir unus omnis Græciæ, philosophiam et musicam egregie didicit. Nam doctus est à Dionysio, qui fuit eximia in musicis gloria. At philosophiæ præceptorem habuit Lysin Pythagoreum, neque prius eum à se dimisit, quam doctrinis tanto antecessit alios, ut facilè intelligi posset, pari modo superaturum omnes in cæteris artibus.

Corn. Nep. vit. Epam. sub initio.

T O

HIS ROYAL HIGHNESS

W I L L I A M

DUKE OF CUMBERLAND,

This Philoſophical Treatiſe,

For a laſting Teſtimony of Gratitude,

Is humbly offered and dedicated,

By His ROYAL HIGHNESS'S

moſt devoted and

moſt dutiful ſervant

ROBERT SMITH

THE PREFACE

TO THE FIRST EDITION.

THE want of an elementary treatife of harmonics, fuch as might properly have been quoted in fupport of my demonftrations, has obliged me to begin the following work from the firft principles of the fcience.

The antient theorifts confidered no other confonances than fuch as are perfect, and yet all their mufical fcales compofed of thefe confonances, have in practice been found difagreeable. The reafon is, they neceffarily contain fome imperfect concords, whofe imperfections are too grofs for the ear to bear with.

The skill of the moderns has been chiefly employed in the bufinefs of tempering the antient fcales, that is, in diftributing thofe groffer imperfections in fome of the concords, among all the reft or the greater part of them. By which means, though the number of imperfect concords be greatly increaſed, yet if their ſeveral imperfections be but as much diminiſhed, the ear will be leſs offended than before. Becauſe it is the tranſition from a better harmony to a worſe, which chiefly gives the offence; as is evident to any one that attends to a piece of muſic performed upon an inſtrument badly tuned. It follows then that the inſtrument would be better in tune, if all the conſonances were made as equally harmonious as poſſible, though none of them were perfect.

And if this be the true deſign in tuning an inſtrument, or tempering a ſcale of ſounds, a theoriſt ought to begin with the ſimpleſt caſe; and inquire in the firſt place, whether it be poſſible for two imperfect conſonances to be made equally harmonious; and if ſo, what muſt be the proportion of their temperaments or imperfections; and alſo whether different conſonances require different proportions. Theſe and the

the like queſtions being rightly ſettled, we may then determine in what proportion thoſe groſſer imperfections in the antient ſcales ought to be diſtributed, ſo as to make all the concords equally harmonious in their kind, either exactly or as near as poſſible.

But as none of the writers that I have ſeen, have attempted to give us the leaſt notion of the nature and conſtitution of imperfect conſonances, nor of any one property or proportion of their effects upon the ear, except a ſingle conjecture whoſe contrary is true (*a*), it was not poſſible for them to determine, from the principles of ſcience, what diſtribution of thoſe groſſer imperfections in the antient ſyſtems, would produce the moſt harmonious ſcale of muſical ſounds.

As this is one of the moſt difficult and important problems in harmonics, in order to a ſcientific ſolution of it I found it neceſſary to premiſe a Theory of Imperfect Conſonances (*b*), wherein

(*a*) Prop. XIII. coroll. 8. (*b*) Sect. VI.

I have demonſtrated as many properties of their Periods, Beats and Harmony as I judged ſufficient for ſolving that problem, and probably any other that belongs to harmonics. This theory with its preliminaries and conſequences takes up a large part of the preſent treatiſe. As to the reſt I chuſe to refer the reader to the book it ſelf or the Index, rather than trouble him with a further account of it: a ſhort one would be imperfect or obſcure, and a perfect one, too long for a preface.

Having been asked more than once, whether an ear for muſic be neceſſary to underſtand harmonics, it may not be amiſs to give this anſwer: That a muſical ear is not neceſſary to underſtand the philoſophy of muſical ſounds; no more than the eye, to underſtand that of colours. Our late Profeſſor of Mathematics was an inſtance of the latter caſe, and the xx[th] propoſition of this treatiſe affords an inſtance of the former. For by the ſolution of that propoſition and a new way of tuning an

VIII[th],

VIIIth, deſcribed in prop. XI, ſchol. 2, art. 6, a perſon of no ear at all for muſic may ſoon learn to tune an organ according to any propoſed temperament of the ſcale; and to any deſired degree of exactneſs, far beyond what the fineſt ear unaſſiſted by theory can poſſibly attain to: and the ſame perſon, if he pleaſes, may alſo learn the reaſon of the practice.

But though an ear for muſic is not neceſſary to underſtand this treatiſe, yet thoſe that are acquainted with muſical ſounds will more readily apprehend many parts of it, and receive more pleaſure from them.

In the firſt ſcholium to prop. XX, I obſerved that the winter ſeaſon had prevented me from tuning an organ by the ſecond table of beats, in order to try what effect the ſyſtem of Equal Harmony might have upon the ear. But upon telling Mr. *Turner*, one of our organiſts at Cambridge, how he might approach near enough to that ſyſtem, by flattening the major III^{ds}, till

till the beats of the vth and vith major with the ſame baſe, went equally ſlow, by his great dexterity and skill in tuning he preſently put my rule in execution upon a ſtop of his organ; and affirmed to me, he never heard ſo fine harmony before, eſpecially in the flat keys; but he added, that for want of more ſounds in every octave the falſe concords were more intolerable than ever: and no wonder, as their common difference from true concords was then increaſed from one fifth to one fourth of the tone.

Nor will it be improper to mention a like experiment made by the accurate hand of Mr. *Harriſon*, well known to the curious in mechanics by his admirable inventions in watch-work and clock-work for keeping time exactly both at ſea and land: which if duly encouraged and purſued will undoubtedly prove of excellent uſe in navigation; by correcting the ſea-charts, with reſpect to longitude, as well as the reckonings of a ſhip, to as great ex-

exactneſs, in all probability, as need be deſired.

But in regard to the experiment I was going to mention, he told me he took a thin ruler equal in length to the ſmalleſt ſtring of his Baſe Viol, and divided the ruler as a monochord, by taking the interval of the major III^d, to that of the VIII^th, as the diameter of a circle, to its circumference. Then by the diviſions on the ruler applied to that ſtring, he adjuſted the frets upon the neck of the viol, and found the harmony of the conſonances ſo extremely fine, that after a very ſmall and gradual lengthening of the other ſtrings, at the nut, by reaſon of their greater ſtiffneſs, he perfectly acquieſced in that manner of placing the frets.

It follows from Mr. *Harriſon*'s aſſumption, that his III^d major is tempered flat by a full fifth of a comma. My III^d determined by theory, upon the principle of making all the concords within the extent of every three octaves as equally harmonious as poſſible, is tem-

tempered flat by one ninth of a comma; or almoſt one eighth, when no more concords are taken into the calculation than what are contained within one octave. That theory is therefore ſupported on one hand by Mr. *Harriſon's* experiment, and on the other by the common practice of muſicians, who make the major III^{d} either perfect or generally ſharper than perfect.

We may gather from the conſtruction of the Baſe Viol, that Mr. *Harriſon* attended chiefly, if not ſolely to the harmony of the conſonances contained within one octave; in which caſe the differences between his and my temperaments of the major III^{d}, VIth and Vth, and their ſeveral dependents, are reſpectively no greater than 4, 3 and 1 fiftieth parts of a comma. And conſidering that any aſſigned differences in the temperaments of a ſyſtem, will have the leaſt effect in altering the harmony of the whole when at the beſt, I think a nearer agreement of that experiment with

with the theory could not be reaſonably expected.

Upon aſking him why he took the interval of the major III^{d} to that of the VIIIth as the diameter to the circumfe-rence of a circle, he anſwered, that a gentleman lately deceaſed had told him it would bring out a very good diviſion of a monochord. Whoever was the au-thor of that hypotheſis, for ſo it muſt be called, as having no connexion with any known property of ſounds, he took the hint, no doubt, from obſerving that as the octave, conſiſting of five mean tones and two limmas, is a little bigger than ſix ſuch tones, or three perfect major III^{ds}, ſo the circumference of a circle is a little bigger than three of its diameters.

When the monochord was divided upon the principle of making the major III^{d} perfect, or but very little ſharper, as in Mr. *Huygens*'s ſyſtem reſulting from the octave divided into 31 equal intervals, Mr. *Harriſon* told me the major VI^{ths} were very bad, and much

worſe

worſe than the v^{ths}. In which he judged rightly, as I further ſatisfied my ſelf by trying the experiment upon an organ; and being ſolicitous to know the reaſon of that effect, that is, why the v^{ths} and VI^{ths} major, when equally tempered, ſhould differ ſo in their harmony, after various attempts I ſatisfied my curioſity.

With a view to ſome other inquiries I will conclude with the following obſervation. That, as almoſt all ſorts of ſubſtances are perpetually ſubject to very minute vibrating motions, and all our ſenſes and faculties ſeem chiefly to depend upon ſuch motions excited in the proper organs, either by outward objects or the power of the Will, there is reaſon to expect, that the theory of vibrations here given will not prove uſeleſs in promoting the philoſophy of other things beſides muſical ſounds.

ROB. SMITH.

Trinity College,
Cambridge, Dec. 31. 1748.

THE PREFACE

TO THE SECOND EDITION.

IN this ſecond edition of theſe harmonics, beſides many ſmaller improvements, the properties of the periods, beats and harmony of imperfect conſonances are more explicitly demonſtrated (a) *and confirmed by very eaſy experiments* (b). *The ultimate ratios of the periods and beats, which are generally more uſeful and elegant than the exact ratios, are proved to be ſufficiently accurate for moſt purpoſes in harmonics* (c). *More methods are added for finding the pitch of an organ* (d) *and for tuning it, either by eſtimation and judgment of the ear* (e), *or more exactly and readily by iſochronous beats of different concords* (f), *as well as by complete*

(a) Lemma to prop. IX, and prop. IX, XI and corollaries.
(b) Prop. XI. ſchol. 2.
(c) Prop. XI. ſchol. 1.
(d) Prop. XVIII. and ſchol. &c.
(e) Sect. IX. art. 1.
(f) Prop. XX. ſchol. 2.

complete tables of beats. An enquiry is made whether coincident pulses be necessary, or only accidental to a perfect consonance (g).

And lastly, as the harpsichord has neither strings nor keys for any of these sounds D^*, A^*, E^*, B^*, F^{**}, A^b, D^b, G^b, *&c, which yet are so often wanted that far the greater part of the best compositions cannot be performed without them, except by substituting for them* E^b, B^b, F, C, G, G^*, C^*, F^*, *&c, respectively, which by differing from them by near a fifth part of the tone, make very bad harmony; and as the old expedient for introducing some of those sounds by inserting more keys in every octave, is quite laid aside by reason of the difficulty in playing upon them; I have therefore invented a better expedient, by causing the several keys of those substitutes,* E^b, B^b, F, C, G, G^*, C^*, F^*, *&c, to strike either* E^b *or* D^*, B^b *or* A^*, F *or* E^*, C *or* B^*,

G

(g) Prop. XI. schol. 4. art. 7. &c.

G *or* F^{**}, G^{*} *or* A^{b}, C^{*} *or* D^{b}, F^{*} *or* G^{b}, *&c.*

For ſince both the ſounds in any one of thoſe couples are ſeldom or never uſed in any one piece of muſic, the muſician by moving a few ſtops before he begins to play it, can immediately introduce that ſound in each couple, which he foreſees is either always or ofteneſt uſed in the piece before him.

Two different conſtruction of thoſe ſtops are here deſcribed (*b*), *one of which is applicable at a ſmall expenſe to any harpſichord ready made, and the other to a new harpſichord, and upon putting them both in practice, they have perfectly anſwered my expectation.*

Several properties and advantages of this changeable ſcale are deſcribed in the eighth Section. In a word, the very worſt keys in the common defective ſcale, by changing a few ſounds are preſently made as complete as the beſt in that ſcale, and more harmonious too, becauſe the

a change-

(*b*) Sect. VIII. art. 18, 19.

changeable ſcale admits of the very beſt temperament, and, which is another advantage, will therefore ſtand longer in tune than the common ſcale which cannot admit that temperament.

Theſe improvements of the harpſichord, it is hoped, may encourage others to apply the like methods to the ſcale of the organ, which is equally capable of them and to greater advantages.

ROB. SMITH,

Trinity College, Cambridge,
Octob. 21, 1758.

A TABLE of the SECTIONS.

ADVERTISEMENT.

HARMONICS.

SECTION I.

Philoſophical Principles of Harmonics.

1. SOUND is cauſed by the vibrations of elaſtic bodies, which communicate the like vibrations to the air, and theſe the like again to our organs of hearing.

Philoſophers are agreed in this, becauſe ſounding bodies communicate tremors to diſtant bodies. For inſtance, the vibrating motion of a muſical ſtring puts others in motion, whoſe tenſion and quantity of matter diſpoſe their vibrations to keep time with the pulſes of air, propagated from the ſtring that was ſtruck. *Galileo* explains this phænomenon by obſerving, that a heavy pendulum may be put in motion by the leaſt breath of the mouth, provided the blaſts be often repeated and keep time exactly with the vibrations of the pendulum; and alſo by the like art in raiſing a large bell; and probably he was the firſt that rightly explained that phænomenon (*a*).

2. If

(*a*) For he ſays, in the perſon of another, il problema poi trito delle due corde teſe all uniſono, che al ſuono dell' una

2. If the vibrations be iſochronous the ſound is called Muſical, and is ſaid to continue at the ſame Pitch; and to be Acuter, Sharper or Higher than any other ſound whoſe vibrations are ſlower; and Graver, Flatter or Lower (*b*) than any other whoſe vibrations are quicker.

For while a muſical ſtring vibrates, if its tenſion be increaſed or its length be diminiſhed, its vibrations will be accelerated; and experience ſhews that its ſound is altered from what is called a graver to an acuter; and on the contrary. And the like alteration of the pitch of the ſound will follow, when the ſame tenſion is given by a weight, firſt to a thicker or a heavier ſtring, and after that to a ſmaller or a lighter of the ſame length, as having leſs matter to be moved by the ſame

una l'altera ſi muova et attualmente riſuona, mi reſta ancora irreſoluto; come anco non ben chiare le forme delle conſonanze et altre particolarità. *Dialogo* 1° *attenente alla Mecanica*, towards the end.

(*b*) As the ideas of acute and high, grave and low, have in nature no neceſſary connexion, it has happened accordingly, as Dr. *Gregory* has obſerved in the preface to his edition of *Euclid*'s works, that the more antient of the Greek Writers looked upon grave ſounds as high, and acute ones as low, and that this connexion was afterwards changed to the contrary by the leſs antient Greeks, and has ſince prevailed univerſally. Probably this latter connexion took its riſe from the formation of the voice in ſinging, which *Ariſtides Quintilianus* thus deſcribes. Γίνεται δὲ ἡ μὲν βαρύτης, κάτωθεν ἀναφερομένου τοῦ πνεύματος, ἡ δ' ὀξύτης, ἐπιπολῆς προϊεμένου. p. 8. Et quidem gravitas fit, ſi ex inferiore parte (gutturis) ſpiritus ſurſum feratur, acumen vero, ſi per ſummam partem prorumpat, as *Meibomius* tranſlates it in his notes. pag. 208.

ſame force of tenſion. And theſe changes in the pitch of the ſound are found to be conſtantly greater or leſſer, according as the length, tenſion, thickneſs or denſity of the ſtring is more or leſs altered (*c*).

3. Therefore if ſeveral ſtrings, however different in length, thickneſs, denſity and tenſion, or other ſounding bodies vibrate all together in equal times, their ſounds will all have one and the ſame pitch, however they may differ in loudneſs or other qualities, and are therefore called Uniſons: and on the contrary, the vibrations of uniſons are iſochronous.

This obſervation reduces the theory of all ſorts of muſical ſounds to that of the ſounds of a ſingle ſtring; I mean with reſpect to their gravity and acuteneſs, which is the principal ſubject of Harmonics (*d*).

4. Con-

(*c*) The Greek muſicians rightly deſcribe the difference between the manner of ſinging and talking. They conſidered two motions in the voice, κινήσεις δύο; the one continued and uſed in talking, ἡ μὲν συνεχής τε καὶ λογική, the other diſcrete and uſed in ſinging, ἡ δὲ διαϛηματική τε καὶ μελῳδική. In the continued motion, the voice never reſts at any certain pitch, but waves up and down by inſenſible degrees; and in the diſcrete motion it does the contrary; frequently reſting or ſtaying at certain places, and leaping from one to another by ſenſible intervals: *Euclid*'s Introductio Harmonica, p. 2. I need not obſerve, that in the former caſe, the vibrations of the air are continually accelerated and retarded by turns and by very ſmall degrees, and in the latter by large ones.

(*d*) *Ptolemy* ſays, Ἁρμονικὴ μέν ἐϛὶ δύναμις καταληπτικὴ τῶν ἐν τοῖς ψόφοις, περὶ τὸ ὀξὺ καὶ βαρὺ, διαφορῶν.

4. Consequently the wider and narrower vibrations of a musical string, or of any other body sounding musically, are all isochronous very nearly.

Otherwise, while the vibrations decrease in breadth till they cease, the pitch of the sound could not continue the same; as by the judgment of the ear we perceive it does, if the first vibrations be not too large: in which case the sound is a little acuter at the beginning than afterwards.

5. In like manner, since the pitch of the sound of a string or bell or other vibrating body, does not alter sensibly while the hearer varies his distance from it; it follows that the larger and lesser vibrations of the particles of air, at smaller and greater distances from the sounding body, are all isochronous: and consequently that the little spaces described by the vibrating particles are every where proportional to the celerity and force of their motions, as in a pendulum (*e*). And this difference of force, at different distances from the sounding body, causes a difference in the loudness of the sound, but not in its pitch.

6. It follows also, that the harmony of two or more sounds, according as it is perfect or imperfect when heard at any one distance, will also be perfect or imperfect at any other distance: which

Harmonics is a power apprehending the differences of sounds, with respect to gravity and acuteness.

(*e*) See Newton's Princip. Lib. II. Prop. 47.

which being a known fact in a ring of bells for inſtance, is mentioned here as a confirmation of theſe principles of Harmonics.

7. If two muſical ſtrings have the ſame thickneſs, denſity and tenſion, and differ in length only, (which for the future I ſhall always ſuppoſe,) mathematicians have demonſtrated, that the times of their ſingle vibrations are proportional to their lengths (*f*).

8. Hence if a ſtring of a muſical inſtrument be ſtopt in the middle, and the ſound of the half be compared with the ſound of the whole, we may acquire the idea of the interval of two ſounds, whoſe ſingle vibrations (always meaning the times) are in the ratio of 1 to 2; and by comparing the ſounds of $\frac{2}{3}$, $\frac{3}{4}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{8}{9}$, $\frac{9}{10}$, &c. of the ſtring with the ſound of the whole, we may acquire the ideas of the intervals of two ſounds, whoſe ſingle vibrations are in the ratio of 2 to 3, 3 to 4, 3 to 5, 4 to 5, 5 to 6, 8 to 9, 9 to 10, &c.

9. A Muſical Interval is a quantity of a certain kind (*g*), terminated by a graver and an acuter ſound.

In

(*f*) As a clear and exact demonſtration of this curious Theorem depends upon one or two more, of no ſmall uſe in Harmonics, and requires a little of the finer ſort of geometry, which cannot well be applied in few words, I have therefore reſerved it to the laſt Section of this Treatiſe; which the reader may conſult, or, taking it for granted at preſent, may proceed without interruption; as he likes beſt.

(*g*) See Dr. *Wallis*'s preface to *Porphyry*'s comment on *Ptolemy*'s Harmonics. Oper. Math. vol. III. *Euclid* ſays, an

In a ring of bells, for example, the ſounds of the firſt and ſecond bells, counting either from the biggeſt or the leaſt, terminate a certain interval; thoſe of the firſt and third a greater interval; thoſe of the firſt and fourth a greater ſtill; &c. So that the interval increaſes by degrees, either as the graver of the two ſounds deſcends, or as the acuter aſcends; and within the interval of the ſounds of the biggeſt and leaſt bells, the intervals between the ſounds of all the reſt are contained.

10. Muſical intervals are Meaſures of the Ratios of the times of the ſingle vibrations of the terminating ſounds, or, *cæteris paribus*, of the lengths of the ſounding ſtrings (*h*).

For it is obſervable in the experiments laſt mentioned (*i*) and is univerſally allowed by muſicians, that when the lengths of thoſe ſtrings have

an Interval is τὸ περιεχόμενον ὑπὸ δύο φθόγγων ἀνωμοίων ὀξύτητι καὶ βαρύτητι, what is contained by two ſounds different in gravity and acuteneſs. Introductio Harmonica p. 1. *Ariſtoxenus* defines a muſical ſound thus, φωνῆς πτῶσις ἐπὶ μίαν τάσιν ὁ φθόγγος, ſonus eſt vocis caſus in unam tenſionem; and an interval thus, Διάστημα δέ ἐστι τὸ ὑπὸ δύο φθόγγων ὡρισμένον, μὴ τὴν αὐτὴν τάσιν ἐχόντων; intervallum vero eſt, quod duobus ſonis, non eandem tenſionem habentibus, terminatur. And he adds, that it is τόπος δεκτικὸς φθόγγων, ὀξυτέρων μὲν τῆς βαρυτέρας τῶν ὁριζουσῶν τὸ διάστημα τάσεων, βαρυτέρων δὲ τῆς ὀξυτέρας; a place capable of ſounds, that are acuter than the graver of the two tenſions (tones or ſounds) that terminate the interval, and graver than the acuter of them. pag. 15.

(*h*) Article 7. (*i*) Art. 8.

have the ſame ratio, the interval of their ſounds is the ſame, whatever be their pitch; that if the acuter of the two ſounds be raiſed higher, and conſequently the ratio of the lengths of thoſe ſtrings be increaſed, the interval is increaſed; and on the contrary, if the acuter ſound be depreſſed lower, that the ſaid ratio and interval are diminiſhed, and reduced to nothing when the ſtrings have the ratio of equality whoſe magnitude is nothing.

Plate I. Fig. 1. Now let the times of the ſingle vibrations of the ſtrings *A*, *B*, *C*, *D*, &c, be continual proportionals in any ratio. Then ſince the interval of the ſounds of *A* and *B* is equal to that of *B* and *C*, or of *C* and *D*, &c, by adding equal intervals together and equal ratios together, it follows, that the interval of the ſounds of *A* and *C*, whoſe ratio is duplicate of *A* to *B* or of *B* to *C*, is double the interval of the ſounds of *A* and *B*, or of *B* and *C*; and that the interval of the ſounds of *A* and *D*, whoſe ratio is triplicate of *A* to *B*, is alſo triple the interval of the ſounds of *A* and *B*, or of *B* and *C* or of *C* and *D*. So that the interval of the ſounds of *A* and *C*, is to that of *A* and *D*, as 2 to 3; and the like is evident of any other equimultiples of the propoſed ratios and intervals, whatever be their number and magnitude.

11. Therefore muſical intervals are proportional to the logarithms of the ratios of the ſingle

vibrations of the terminating ſounds, or, *cæteris paribus*, of the lengths of the vibrating ſtrings. Becauſe logarithms are numeral meaſures of ratios; and all ſorts of meaſures, of the ſame magnitudes are proportional to one another (*k*).

12. For brevity ſake the word vibration is often uſed for the time of a complete vibration, which paſſes between the departure of the vibrating body from any aſſigned place and its return to the ſame. Such is the time between the ſucceſſive pulſes of air upon the ear; a pulſe being made while the air is compreſſed and condenſed in its progreſs, but not in its regreſs; it being then relaxed and rarified to a greater degree than the quieſcent air is (*l*). And though the pulſes of ſounds of a different pitch have different durations, they may yet be abſtractly conſidered as if they were inſtantaneous; by taking only the middle inſtant of each pulſe.

(*k*) See Mr. *Cotes*'s Harmonia Menſurarum, pag. 1.

(*l*) See *Newton*'s Principia, Book 2. Prop. 43. Caſ. 1.

SECTION

SECTION II.

Of the Names and Notation of consonances and their intervals.

1. PLATE I. Fig. 2. If a mufical ftring *CO* and its parts *DO*, *EO*, *FO*, *GO*, *AO*, *BO*, *cO*, be in proportion to one another as the numbers 1, $\frac{8}{9}$, $\frac{4}{5}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{8}{15}$, $\frac{1}{2}$, their vibrations will exhibit the fyftem of 8 founds which muficians denote by the letters *C*, *D*, *E*, *F*, *G*, *A*, *B*, *c*.

Fig. 3. And fuppofing thofe ftrings to be ranged like ordinates to a right line *Cc*, and their diftances *CD*, *DE*, *EF*, *FG*, *GA*, *AB*, *BC*, not to be the differences of their lengths, as in fig. 2, but to be of any magnitudes proportional to the intervals of their founds, the received Names of thefe intervals are fhewn in the following Table; and are taken from the numbers of the ftrings or founds in each interval inclufively; as a Second, Third, Fourth, Fifth, &c, with the epithet of *major* or *minor*, according as the name or number belongs to a greater or fmaller total interval; the difference of which refults chiefly from the different magnitudes of the major and minor fecond, called the Tone and Hemitone.

C..

C . . *D* . . *E* . *F* . . *G* . . *A* . . *B* . *c* . .

1 . . $\frac{8}{9}$ . . $\frac{4}{5}$. $\frac{3}{4}$ . . $\frac{2}{3}$ . . $\frac{3}{5}$ . . $\frac{8}{15}$. $\frac{1}{2}$ . .

Perfect Ratios,		Interval's Names,	Marks,	Elements.
C : *c* :: 2 : 1	*C c*	Octave	VIII	3T + 2t + 2H
B : *c* :: 16 : 15	*B c*	Hemitone	H or 2^{d}	
C : *B* :: 15 : 8	*C B*	VII major	VII	3T + 2t + H
C : *D* :: 9 : 8	*CD*	Tone major	T or II	
D : *c* :: 16 : 9	*D c*	7^{th} minor	7^{th}	2T + 2t + 2H
A : *c* :: 6 : 5	*A c*	3^{d} minor	3^{d}	T + H
C : *A* :: 5 : 3	*C A*	VI major	VI	2T + 2t + H
C : *E* :: 5 : 4	*C E*	III major	III	T + t
E : *c* :: 8 : 5	*E c*	6^{th} minor	6^{th}	2T + t + 2H
G : *c* :: 4 : 3	*G c*	4^{th} minor	4^{th}	T + t + H
C : *G* :: 3 : 2	*CG*	V major	V	2T + t + H
F : *B* :: 45 : 32	*F B*	IV major	IV	2T + t
B : *f* :: 64 : 45	*B f*	5^{th} minor	5^{th}	T + t + 2H
D : *E* :: 10 : 9	*DE*	Tone minor	t	
81 : 80		Comma	c	T — t

2. Hence it is, if the ratio of the ſingle vibrations of any two ſounds, or, *cæteris paribus*, of the lengths of two vibrating ſtrings, be any of thoſe in the firſt column of the table, that their interval, and the conſonance too, retains the name in the third column, whether the intermediate ſounds be preſent or abſent.

3. Fig. 3. In the line *Cc* produced beyond *c*, if we take the intervals *D d*, *E e*, *F f*, &c, ſeverally

verally equal to the octave Cc, and make the length of the several strings at d, e, f, &c, equal to half the lengths of those at D, E, F, &c, all the intervals within this higher octave cc', will also consist of major and minor tones and hemitones, ranged in the same order as in the lower octave Cc.

And the names of intervals larger than one or more octaves, are also taken from the number of the strings in them inclusively. Thus the interval Cd is called a Ninth, Ce a Tenth, Cf an Eleventh, Cg a Twelfth, &c, with the epithet of *major* or *minor* as before; and are thus denoted, IX or VIII + II, X or VIII + III, 11th or VIII + 4th, XII or VIII + V, &c, the units in the compound marks being constantly one more than those in the simple ones, because the intermediate string at the end of the octave is counted twice. The same is to be understood in all compounded notations.

4. A Comma is the interval of two sounds whose single vibrations have the ratio of 81 to 80, and is the difference of the major and minor tones (*m*).

5. Any one of the ratios in the first column of the foregoing Table, except 80 to 81, or any one of them compounded once or oftener with the ratio 2 to 1 or 1 to 2, is called a Perfect ratio when reduced to its least terms. And when the times

(*m*) For the ratio of 9 to 8 diminished by the ratio of 10 to 9, is the ratio of 9×9 to 8×10, or of 81 to 80.

times of the ſingle vibrations of any two ſounds have a perfect ratio, the conſonance and its interval too is called Perfect; and is called Imperfect or Tempered when that perfect ratio and interval is a little increaſed or decreaſed.

6. Any ſmall increment or decrement of a perfect interval is called reſpectively the Sharp or Flat Temperament of the imperfect conſonance, and is meaſured moſt conveniently by the proportion it bears to a comma.

7. As the addition and ſubtraction of logarithms anſwers to the multiplication and diviſion of their correſponding Tabular numbers, that is, to the compoſition and reſolution of the ratios of thoſe numbers to an unit; ſo the addition and ſubtraction of muſical intervals anſwers to the compoſition and reſolution of the ratios of the ſingle vibrations of the terminating ſounds, or, *cæteris paribus*, of the lengths of the vibrating ſtrings: and on the contrary.

In the following examples, the compoſition and reſolution of perfect ratios is intimated by the multiplication of their terms, placed, in the form of fractions, upright and inverted, reſpectively.

$$\left\{\begin{array}{l} \text{As } \frac{1}{2} = \frac{8}{15} \times \frac{15}{16} = \frac{9}{16} \times \frac{8}{9} = \frac{3}{5} \times \frac{5}{6} = \\ \text{So VIII} = \text{VII} + 2^{d} = 7^{th} + \text{II} = \text{VI} + 3^{d} = \end{array}\right.$$

$$\begin{array}{l} \frac{5}{8} \times \frac{4}{5} = \frac{2}{3} \times \frac{3}{4} = \frac{8}{9}\times\frac{8}{9}\times\frac{8}{9}\times\frac{9}{10}\times\frac{9}{10}\times\frac{15}{16}\times\frac{15}{16}. \\ 6^{th} + \text{III} = \text{V} + 4^{th} = 3\,\text{T} + 2\,\text{t} + 2\,\text{H} \quad (n). \end{array}$$

$$\left\{\begin{array}{l|l|l} \text{As } \frac{3}{5} = \frac{3}{4} \times \frac{4}{5} & \frac{5}{8} = \frac{3}{4} \times \frac{5}{6} & \frac{2}{3} = \frac{4}{5} \times \frac{5}{6} \\ \text{So VI} = 4^{th} + \text{III} & 6^{th} = 4^{th} + 3^{d} & \text{V} = \text{III} + 3^{d} \end{array}\right.$$

$$\begin{array}{l|l|l} \text{As } \frac{3}{4} = \frac{4}{5} \times \frac{15}{16} & \frac{3}{4} = \frac{5}{6} \times \frac{9}{10} & \frac{4}{5} = \frac{8}{9} \times \frac{9}{10} \\ \text{So } 4^{th} = \text{III} + 2^{d} & 4^{th} = 3^{d} + \text{t} & \text{III} = \text{T} + \text{t} \end{array}$$

$$\left\{\begin{array}{l|l|l} \text{As } \frac{5}{6} = \frac{8}{9} \times \frac{15}{16} & \frac{3}{5} \times \frac{3}{2} = \frac{9}{10} & \frac{2}{3} \times \frac{4}{3} = \frac{8}{9} \\ \text{So } 3^{d} = \text{T} + \text{H} & \text{VI} - \text{V} = \text{t} & \text{V} - 4^{th} = \text{T} \end{array}\right.$$

$$\begin{array}{l|l|l} \text{As } \frac{3}{4} \times \frac{6}{5} = \frac{9}{10} & \frac{3}{4} \times \frac{5}{4} = \frac{15}{16} & \frac{8}{9} \times \frac{10}{9} = \frac{80}{81} \\ \text{So } 4^{th} - 3^{d} = \text{t} & 4^{th} - \text{III} = \text{H} & \text{T} - \text{t} = \text{c}. \end{array}$$

8. Hence the hemitones, and tones major and minor, being the differences of the intervals, III, 4th, V, VI, and of their compliments to the octave, may be confidered as the Elements that compound the intervals of all perfect concords,

(n) See the Column of Elements in the foregoing Table.

cords, as in the laſt column of the former Table compared with Fig. 3. So that the leaſt intervals in a muſical ſcale are founded upon the harmony of the concords (*o*).

SECTION III.

Of perfect conſonances and the Order of their ſimplicity.

1. PLATE I. Fig. 4. When a ſingle ſound is heard, the ſeries of equal times between the ſucceſſive pulſes of air that beat on the ear (*p*), may be repreſented by a ſeries of equal parts contained in a right line; as in 02, 03, 04, &c. Conſequently when two ſounds are heard, two of thoſe lines, as 02 and 03, will rightly repreſent the two ſeries of equal times, if the magnitude of the equal parts in one line, be to the magnitude of thoſe in the other, in the ratio of the ſingle vibrations of the ſounds: or, the whole lines being ſuppoſed equal, if the numbers of aliquot parts in each, as 2 and 3, be ſeverally the ſame as the leaſt numbers of the vibra-

(*o*) The old method of reſolving concords into their elements may be ſeen in Dr. *Wallis*'s diviſion of the monochord, or ſection of the muſical Canon, as the antients called it. Philoſoph. Tranſact. N°. 238. or Abridg. by Lowthorp. vol. 1. p. 698. firſt edit.

(*p*) Sect. I. Art. 12.

vibrations of each ſound, made in the ſame time repreſented by the line o2 or o3 (*q*).

2. And the ſounds being heard together, if we conceive the two equal and parallel lines that rightly repreſent them, as o2 and o3, to coincide throughout, the points that divide the ſeparate lines, will ſubdivide the combined lines into ſmaller portions, as in Fig. 5, repreſenting a third ſeries or Cycle of times, in which the pulſes of both ſounds interchangeably ſucceed one another in beating upon the ear.

3. Such a mixture of pulſes, ſucceeding one another in a given cycle of Times, terminated at both ends by coincident pulſes, and ſufficiently repeated, is the phyſical cauſe that excites the ſenſation of a given conſonance: Eſpecially when conſidered as diſtinct from any other conſonance, whoſe ſingle vibrations having a different ratio from that of the former, will conſtitute a different cycle, and excite a different ſenſation. But if that ratio be the ſame, though the abſolute times be different, the conſonances are ſimilar and may be looked upon as the ſame in this reſpect, that their cycles have the ſame form; the times in both having the ſame order, and the ſame proportions; and in this other alſo, that the interval of the ſounds is the ſame (*r*).

4. This being premiſed, one conſonance may be conſidered as more or leſs ſimple than another, ac-

(*q*) See Art. 12. following.

(*r*) Art. 10. Sect. 1.

according as the cycle of times belonging to it, is more or leſs ſimple than the cycle belonging to the other. And upon this principle all conſonances may be ranged in due Order of ſuch ſimplicity, by the help of the following Rule.

5. *One Conſonance is Simpler than another in the ſame Order, as the ſum of the leaſt terms, expreſſing the ratio of the ſingle vibrations, is ſmaller than the like ſum in the other conſonance; and when ſeveral ſuch ſums are the ſame, theſe conſonances are ſimpler in the ſame order, as the leſſer terms of their ratios are ſmaller.*

For the ſimplicity of a conſonance or cycle of times, conſiſts partly in the number of times contained in the cycle, and partly in the different proportions they bear to one another.

Fig. 4. When the numbers of times in different cycles are different, and the times in each cycle are equal to one another, as when we combine the ſounds 01 and 01, 01 and 02, 01 and 03, 01 and 04, 01 and 05, &c, the cycles of this ſort may be ranged in the order of their ſimplicity above defined, by the order of the numbers of equal times in the cycles, or of the magnitudes of the numbers 1, 2, 3, 4, 5, 6, &c, or of 2, 3, 4, 5, 6, 7, &c, that is, of the ſums of the terms of the ratios 1 to 1, 1 to 2, 1 to 3, 1 to 4, 1 to 5, &c.

In the other caſe, where the numbers of times in different cycles are the ſame, and the times in each cycle bear different proportions to one another, as when we combine the ſounds 01 and

06,

o6, o2 and o5, o3 and o4, that cycle is ſimpler than another, in which the equal times between the pulſes of the acuter ſound, are leſs interrupted and ſubdivided by the pulſes of the graver.

Accordingly in the firſt of theſe cycles compoſed of o1 and o6, not one of the 6 equal times between the pulſes of the acuter ſound o6, is ſubdivided by any pulſe of the graver o1; but in the ſecond cycle compoſed of o2 and o5, one of the 5 equal times, between the pulſes of the acuter ſound o5, is ſubdivided by one pulſe of the graver o2; and in the third cycle compoſed of o3 and o4, two of the 4 equal times in the acuter ſound o4, are ſubdivided by 2 pulſes of the graver o3. By which it appears, that the firſt cycle is ſimpler than the ſecond, and the ſecond ſimpler than the third; and that the order of ſimplicity of this ſort of cycles, anſwers to the order of the magnitudes 1, 2, 3 of the leſſer terms of the ratios.

6. Now by the firſt part of the rule above, the integers in the ſecond column of the following table, are the ſeveral ſums of the terms of the oppoſite ratios in the firſt, diminiſhed by 1, which alters not the order of their magnitudes, but only makes the ſeries begin with 1, anſwering to the ſimpleſt conſonance.

By the ſecond part of the rule, the ratios whoſe terms have the ſame ſum, as 1 : 6, 2 : 5, 3 : 4, are ranged in the order of their leſſer terms 1, 2, 3, or, which alters not the order, of thoſe terms ſeverally diminiſhed by 1, as of 0, 1, 2, or of the

 fractions

A table of the Order of the ſimplicity of conſonances of two ſounds.

Ratios of the vibrations.	Order of the ſimplicity.	Intervals of the ſounds.	Continuation of the table.		
1 : 1	1	o	1 : 15	15	3VIII + VII
1 : 2	2	VIII	1 : 16	16	4VIII
1 : 3	3	VIII + V	2 : 15	$16\frac{1}{8}$	2VIII + VII
1 : 4	4	2VIII	5 : 12	$16\frac{1}{2}$	VIII + 3d
2 : 3	$4\frac{1}{2}$	V	8 : 9	$16\frac{7}{8}$	T
1 : 5	5	2VIII + III	1 : 18	18	4VIII + T
1 : 6	6	2VIII + V	3 : 16	$18\frac{2}{9}$	2VIII + 4th
2 : 5	$6\frac{1}{3}$	VIII + III	4 : 15	$18\frac{1}{3}$	VIII + VII
3 : 4	$6\frac{2}{3}$	4th	9 : 10	$18\frac{8}{9}$	t
1 : 7	7		1 : 20	20	4VIII + III
3 : 5	$7\frac{2}{3}$	VI	5 : 16	$20\frac{2}{5}$	VIII + 6th
1 : 8	8	3VIII	1 : 22	22	
4 : 5	$8\frac{3}{4}$	III	3 : 20	$22\frac{2}{11}$	2VIII + VI
1 : 9	9	3VIII + T	5 : 18	$22\frac{4}{11}$	VIII + 7th
1 : 10	10	3VIII + III	8 : 15	$22\frac{7}{11}$	VII
2 : 9	$10\frac{1}{5}$	2VIII + T	1 : 24	24	4VIII + V
3 : 8	$10\frac{2}{5}$	VIII + 4th	9 : 16	$4\frac{2}{3}$	7th
5 : 6	$10\frac{4}{5}$	3d	1 : 28	28	
1 : 12	12	3VIII + V	5 : 24	$28\frac{2}{7}$	2VIII + 3d
3 : 10	$12\frac{1}{3}$	VIII + VI	9 : 20	$28\frac{4}{7}$	VIII + t
4 : 9	$12\frac{1}{2}$	VIII + T	1 : 30	30	4VIII + VII
5 : 8	$12\frac{2}{3}$	6th	15 : 16	$30\frac{14}{15}$	H
6 : 7	$12\frac{5}{6}$		32 : 45	$76\frac{31}{38}$	IV
			45 : 64	$108\frac{22}{27}$	5th

fractions $\frac{0}{3}$, $\frac{1}{3}$, $\frac{2}{3}$, whoſe common denominator 3 is the number of the ratios whoſe terms have the ſame ſum 7. Theſe fractions either by themſelves or the mixt numbers 6, $6\frac{1}{3}$, $6\frac{2}{3}$, made by annexing them to the number 6, may therefore denote the order of the Modes of ſimplicity of ſuch conſonances as have the ſame Degree of ſimplicity denoted by 6 or $7-1$. And thus the order of the ſimplicity of all conſonances whatever, is denoted by the order of the magnitudes of the integers and mixt numbers in the ſecond column of the table.

7. This ſeries increaſes from unity in ſeveral arithmetical progreſſions, except that a term or two is here and there omitted, where ratios occur which being reducible to ſimpler terms, have been conſidered before, or elſe are not Perfect Ratios, which are ſuch only whoſe terms are 1, 2, 3, 5, with their powers and products (*s*).

For example, writing down all the ratios in due order, whoſe terms make a given ſum, as 1 to 8, 2 to 7, 3 to 6, 4 to 5, I reject the two middlemoſt for the reaſons juſt mentioned, and place the reſt in the firſt column of the table; which may thus be continued with certainty and order as far as we pleaſe.

8. Hence we may diſtinguiſh conſonances into two ſorts, Pure and Interrupted; pure, where none of the equal times between the pulſes of the acuter ſound, is ſubdivided by any intermediate

(*s*) Sect. II. art. 5.

diate pulſe of the graver; and interrupted, when any of thoſe equal times are interrupted by one or more pulſes of the graver ſound.

In the ſecond column of the table, the leaſt ſimple or loweſt mode of each degree of interrupted conſonancy, is every where placed above the next inferior degree of pure conſonancy, as $4\frac{1}{2}$ above 5.

For ſhould we depreſs the mode $4\frac{1}{2}$ to a place next below the degree 5, why not even to a place next below 6? though not below $6\frac{1}{3}$, as being a more complex mode of a leſs ſimple degree. But if that were allowable, by parity of reaſon we ought to depreſs $4\frac{1}{2}$, $6\frac{1}{3}$, $6\frac{2}{3}$, next below 7, though not below $7\frac{2}{3}$, and likewiſe $4\frac{1}{2}$, $6\frac{1}{3}$, $6\frac{2}{3}$, $7\frac{2}{3}$, below 8, though not below $8\frac{3}{4}$, and alſo $4\frac{1}{2}$, $6\frac{1}{3}$, $6\frac{2}{3}$, $7\frac{2}{3}$, $8\frac{3}{4}$ below 9 and 10, and ſo forth to infinity: which, by depreſſing all the modes of interrupted conſonancy, below all the degrees of pure conſonancy, would render them heterogeneal, and incapable of any order or compariſon with one another. The table is therefore rightly ordered.

9. Hitherto we have only conſidered the number and proportions of the times in the cycle by which a conſonance is repreſented, without regard to the quality of the pulſes, as to magnitude, duration, ſtrength, weakneſs or other accidents; whereas the pulſes of graver ſounds are generally ſtronger, larger, obtuſer, and of longer duration than thoſe of acuter ſounds, and affect the ear differently. But ſtill this alters not the rational idea of the conſonance, as above deſcribed, provided

provided we take the middle inſtant of each pulſe as we did in Art. 1; nor does the ear perceive any alteration in the kind of mixture, or in the interval upon ſoftening or ſwelling either ſound, while the other retains the ſame ſtrength.

10. It is well known in general, that ſimpler conſonances affect the ear with a ſmoother and pleaſanter ſenſation, and the leſs ſimple with a rougher and leſs pleaſant one. And this analogy ſeems to hold true according to the order in the table, as far as the ear can judge with certainty. Thoſe that are willing to try the experiment, may readily do it by the help of the third column of the table, ſhewing the muſical intervals anſwering to the reſpective conſonances. But the analogy will be plainer perceived by intermitting ſeveral conſonances, and trying it, for example, in this ſeries of all the concords not exceeding the octave; VIII, V, 4th, VI, III, 3d, 6th; but then they ſhould not be tempered as uſual, but tuned perfect. And if the experimenter be ſkilful in melody and compoſition, he muſt endeavour, as much as poſſible, to diveſt himſelf of all habitual prepoſſeſſions in favour of this or that concord, or ſucceſſion of concords, acquired from the rules and practice of his art; in order to an impartial judgment of the ſimple perception of the ſmoothneſs and ſweetneſs of each concord, and a fair compariſon of ſuch perceptions only.

11. Though nature has appointed no certain limit between concords and diſcords, yet as muſicians diſtinguiſh conſonances by thoſe names for

their own uſes, I may do the like for mine; calling uniſons, III^{ds}, V^{ths} and VI^{ths}, and their complements to the VIIIth and compounds with VIII^{ths}, Concords, and all other conſonances, Diſcords.

12. If the times of the ſingle vibrations of any two ſounds be V and v, and if $V : v :: R : r$, repreſenting the leaſt integers in that ratio; the length of the cycle of times between the ſucceſſive coincidences of the pulſes of V and v, is $rV = Rv$. Becauſe theſe multiples of V and v are the leaſt of any that can be equal.

For the ſame reaſon, if $V : x :: S : s$ in the leaſt integers, the cycle $sV = Sx$.

13. Hence the length of the cycle of V and v, is to that of V and x, as r to s; that is, the cycles of conſonances that have a common ſound or vibration V, are proportional to the Numerators of the fractions $\frac{r}{R}V = v$, $\frac{s}{S}V = x$, expreſſing the times of the ſingle vibrations of the other ſounds, as in Fig. 3, or to the leſſer terms of the ratios in the firſt column of the table of the order of the ſimplicity of conſonances.

14. Conſequently were the degrees of ſimplicity of conſonances to be eſtimated by the frequency of the coincidences of their pulſes, or the ſhortneſs of their cycles, as is commonly ſuppoſed; the uniſons, VIII^{ths}, VIII + V^{ths}, 2 VIII^{ths}, 2 VIII + III^{ds}, &c, whoſe cycles are but 1 vibration of the baſe, would be equally ſimple; and the ſame may be ſaid of the V^{ths}, VIII + III^{ds}, 2 VIII + T^{s}, &c, whoſe ſeveral cycles are but 2 vibra-

vibrations of the baſe; and the ſame alſo of all conſonances having the ſame number for the leſſer term of their perfect ratios; which ſhews that the frequency of coincidences is, of itſelf, too general a character of the ſimplicity or ſmoothneſs of a conſonance, and therefore an imperfect one.

SECTION IV.

Of the antient Syſtems of perfect conſonances.

1. IF no other primes but 1, 2, 3 were admitted to the compoſition of perfect ratios, a ſyſtem of ſounds thence reſulting could have no perfect thirds; nor any perfect conſonance whoſe vibrations are in any ratio having the number 5, or any multiple of it, for either of its terms, as 5 to 4, 6 to 5, 10 to 9, 16 to 15, &c: it being impoſſible for any powers and products of the given primes 1, 2, 3 to compoſe any other prime or multiple of it.

2. Fig. 3. The minor tones *DE*, *GA* being thus excluded, and major tones being put in their places, every perfect major III^d will be increaſed by a comma, as being the difference of the tones (*t*); and every hemitone and perfect minor 3^d will be as much diminiſhed; becauſe the 4^ths and

(*t*) Sect. II. Art. 4.

and v^{ths}, as *CF* and *Fc*, *cG* and *GC*, are perfect, whether 5 be admitted or not, as depending on the primes 1, 2, 3, only.

3. These diminished hemitones being called Limmas, the octave is now divided into 5 major tones and 2 limmas; as represented to the eye in Plate II. Fig. 6; where the consonances whose vibrations are expressed by such high terms as the powers of 8 and 9, &c, must needs be disagreeable to the ear, according to the foregoing analogy between the agreeable smoothness of a consonance and the simplicity of the numbers expressing the ratio of its vibrations (*u*): and that in reality they are so, any one will soon find if he pleases to try the following experiment.

4. Fig. 6. Ascending by a perfect vth and descending by a perfect 4th alternately, upon an organ or harpsichord tune the following sounds, from *F* to *C*, *C* to *G*, *G* to *D*, *D* to *A*, *A* to *E*, *E* to *B*, and the octave *Ff* will then be divided into 5 major tones and 2 limmas; because the differences of those successive v^{ths} and 4^{ths} are major tones.

Then having tuned perfect octaves to every one of those notes, try the consonances that would be perfect if the number 5 were admitted, as thirds major and minor, with their complements to the VIIIth and compounds with VIII^{ths}; and you will find them extremely disagreeable (*x*).

5. But

(*u*) Sect. III. Art. 10.

(*x*) The v^{ths} and 4^{ths} being tuned by the judgment of the ear, if any one doubts whether their single vibrations be

5. But if 5 be admitted among the muſical primes, the ratios 10 to 9 and 16 to 15, belonging to the minor tone and the hemitone, are alſo admitted, and the elements that now compoſe the octave, are 3 major tones, 2 minor and 2 hemitones, as in Fig. 3.

PROPOSITION I.

A ſyſtem of ſounds whoſe elements or ſmalleſt intervals are tones major and minor and hemitones, will neceſſarily contain ſome imperfect concords.

6. Whatever be the order of thoſe elements in any one octave, it muſt be the ſame in every one; to the end that every ſound may have a perfect octave to it, as being the beſt concord. And in order to have as many perfect v^ths^ as poſſible, and conſequently VIII + v^ths^, which concords are the ſecond beſt (*y*), the elements muſt be ranged in ſuch order, that the contiguous couples ſhall make as many perfect thirds as poſſible, both major

be as 3 to 2 and 3 to 4, let the Muſician compare the ſound of $\frac{2}{3}$ of a muſical ſtring, and alſo $\frac{3}{4}$ of it with that of the whole, and he will acknowledge theſe concords and thoſe which he tuned upon the inſtrument to be the ſame, and of conſequence to have the ſame ratios of their ſingle vibrations.

(*y*) See the table of the order of concords in Sect. III. Art. 5.

jor and minor; theſe being the intervals which compoſe the perfect v[ths]. And that order being rightly determined, we ſhall have the greateſt number of perfect concords of all ſorts. Becauſe the complements to the octave, of perfect thirds and v[ths], will alſo be perfect, and ſo will their compounds with any number of VIII[ths].

Now it is obſervable of the ſeven elements T, T, T, *t*, *t*, H, H, which compoſe an octave, that T and H, T and *t* are the only couples which make perfect thirds (*z*), all the reſt, T and T, *t* and *t*, *t* and H, H and H, making thirds imperfect by a comma, except H and H, which compoſe an imperfect tone, bigger than the major tone by almoſt a comma (*a*).

Hence either T and H, or T and *t* muſt be the outermoſt elements in the octave, as in the following table.

For if the firſt element in every octave in the ſyſtem be T and the ſeventh be H, the ſeventh in any octave, combined with the firſt in the next octave, will compoſe the interval H + T of a perfect

(*z*) Sect. II. Art. 5 and 7.

(*a*) Putting $H = \log. \frac{16}{15} = 0.02803$

$$\text{Then } 2H = 2 \times \log. \frac{16}{15} = 0.05606$$

$$\text{And } T = \log. \frac{9}{8} = 0.05115$$

$$\text{Whence } 2H - T = 0.00491$$

$$\text{And the Comma} = \log. \frac{81}{80} = 0.00540$$

$$\text{Difference } \quad 0.00049$$

perfect minor 3^d^, and thus the contiguous octaves will be joined in perfect concord.

Table of the Elements.

&c	7	1 2	3 4 5	6 7	1 &c
			H+T+t		
Caf. 1.	H	T+t		T+H	T
			t+T+H		
	H	T+H	t+T+t	T+H	T
Caf. 2.	t	T+t	H+T+H	T+t	T
			t+T+H		
	t	T+H		T+t	T
			H+T+t		

Likewife if the firft element in every octave be T and the feventh be *t*, here alfo the feventh in any octave, together with the firft in the next octave, will compofe the interval *t*+T of a perfect major III^d^, and thus the contiguous octaves will again be joined in perfect concord; and in no other cafe befides thofe two, as appears by the obfervation above.

Caf. 1. Now if the fecond element be *t*, the firft joined to it compofes the perfect major III^d^, T+*t*. And if the fixth element be T, the feventh joined to it will compofe the perfect minor third T+H.

Two of the feven elements in the octave being thus difpofed of at each end of it, the contiguous couples

couples of the remaining three cannot compose perfect thirds in any order different from this, $H+T+t$, or its reverse $t+T+H$; both which being transferred into the interval between those extreme couples, shew, that the elements in the second and third places, compose either the imperfect minor third $t+H$, or the imperfect major third $t+t$.

If H be the second element, as in the third rank of the table, the first couple does now compose the perfect minor third $T+H$, and the last being also $T+H$, as before, the three remaining elements must have this order $t+T+t$, to make perfect thirds of their contiguous couples; and being thus transferred into the interval between those extreme couples, they shew, that the second and third elements do again compose an imperfect minor third $H+t$.

Cas. 2. Here also the sixth element must be T, since no other joined to the seventh can make a perfect third, as $T+t$.

Now if the second element be t, this joined to the first makes the perfect major third $T+t$. And two of the seven elements in the octave being thus joined at each end of it, the contiguous couples of the remaining three, cannot compose the intervals of perfect thirds in any order different from this, $H+T+H$; which being transferred into the interval between the extreme couples, shews, that the second and third elements do here also compose the interval $t+H$ of an imperfect minor third.

If

If H be the ſecond element, as in the next lower ranks, then the firſt couple compoſe the interval T+H of a perfect minor 3^d, and the laſt couple being T+*t* as before, the three remaining elements muſt have this order, *t*+T+H, or its reverſe, H+T+*t*, for the reaſon above; and being thus transferred into the middle interval, they ſhew, that the elements in the ſecond and third places do again compoſe an imperfect minor third, H+*t*, or elſe an imperfect tone H+H; which being joined to the major tone on either ſide of it, compoſes an imperfect major third, greater than *t*+T by almoſt two commas, as appears by the preliminary obſervation.

Now any one of thoſe imperfect minor thirds, *t*+H, together with the contiguous perfect major III^d, compoſes a fifth equally imperfect, and ſo does the imperfect major third *t*+*t* with the perfect minor third next to it. And the complements to the VIII^th of theſe imperfect concords, as well as their compounds with VIII^ths, are alſo equally imperfect, which proves the propoſition. For having ſhewn the neceſſary defects in thoſe ſix arrangements of the ſeven elements, we are freed from the trouble of conſidering the reſt (*b*). Q. E. D.

7. *Coroll.* Of thoſe ſix arrangements of the elements, the firſt and fifth in the table are equally good

(*b*) Mr. *De Moivre*'s general corollary to the XVI problem of his Doctrine of Chances, gives 210 permutations of theſe ſeven things, T, T, T, t, t, H, H.

good, and better than any one of the rest, as producing as many perfect thirds, and a greater number of perfect v^ths.

Pl. II. Fig. 7. In order to enumerate them with certainty and ease, if the circumference of a circle, be divided into seven arches, *CD*, *DE*, *EF*, *FG*, *GA*, *AB*, *BC*, proportional to T, *t*, H, T, *t*, T, H, placed in the respective angles at the center; they and their sums, whether smaller or greater than the circumference, here considered as a continued spiral, will represent all the intervals in a system composed of any number of octaves, and the corresponding intervals in different octaves will be denoted by the same arch and letters: as appears by conceiving the base of the third Figure coiled round into the circumference of a circle, equal to the line *Cc* or *cc'* &c. (*c*)

In this notation then we have only three major III^ds, *CE*, *FA*, *GB*, and they all perfect; and four minor thirds, *DF*, *EG*, *AC*, *BD*, the first of which being composed of $t+$H, instead of T+H,

(*c*) In this notation of intervals by circular arches, that the reader may not be at a loss for a suitable notation of the lengths of the corresponding homogeneal strings; let the radius *OC* be 1 and in *OD*, *OE*, *OF*, *OG*, *OA*, *OB*, *OC*, from the center set off $\frac{8}{9}$, $\frac{4}{5}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{8}{15}$, $\frac{1}{2}$ of the radius. These are the same lengths as those of the Monochord in Fig. 2, or Fig. 3; and as a regular curve drawn thro' the ends of the parallel strings in Fig. 3. is a Logistic Line whose Asymtote is the line *Cc*, so a regular curve drawn thro' the ends of the diverging strings in Fig. 7. is an Equiangular Spiral whose Pole is the center of the circle. See Sect. I. Art. 10. and Mr. *Cotes*'s Harmonia Mensurarum, Prop. V and VI.

T+H, is too ſmall by a comma; and ſix fifths, *FAC*, *CEG*, *GBD*, *DFA*, *ACE*, *EGB*, all perfect but *DFA*, which being compoſed of the defective minor third *DF* and the perfect major IIId *FA*, is too ſmall by a comma.

Theſe imperfections being cauſed by the contiguity of *t* and H in the cycle of the elements, cannot be avoided while the hemitones are ſeperated; there being but 3 major tones in the cycle; and if they be joined, as in Fig. 12, the conſequences will be worſe.

The reſt will appear by enumerating the thirds and fifths in the 8th, 9th, 10th, 11th, and 12th Figures, made according to the other five arrangements in the Table of Elements (*d*).

8. Now if any one pleaſes to try the following experiment, he will find what effect theſe imperfect fifths and fourths and their compounds with VIIIths, will have upon the ear; that of the thirds and ſixths having been tried before (*e*).

In

(*d*) Sir *Iſaac Newton* happily diſcovered, (Optics Book 1, Part 2, Prop. 3) that the breadths of the ſeven primary colours in the ſun's image, produced by the refraction of his rays through a priſm, are proportional to the ſeven differences of the lengths of the eight muſical ſtrings, *D*, *E*, *F*, *G*, *A*, *B*, *C*, *d*, when the intervals of their ſounds are T, H, t, T, t, H, T: which order is remarkably regular; but though it agrees beſt with the priſmatic colours, it is not the propereſt for a ſyſtem of concords, as producing one major third, two minor thirds and two fifths ſeverally imperfect by a comma. See Fig. 13. N°. 2.

(*e*) Sect. IV. Art. 4.

In Fig. 3, tune upwards from *C* the two perfect v^{ths} *CG*, *Gd*, and the perfect XVIIth, or 2VIII+III, *Cé*, then downwards the vth *éa*, and the intermediate fifth *ad* will be too little by a comma, as including the imperfect minor third *df*. And by tuning an eighth below *a* we have the imperfect fourth *Ad* too large by a comma.

9. The disagreeable effect of this fifth *da* and fourth *dA* in every octave, and of their compounds with VIII^{ths}, and also of the third *df* and and sixth *fd'* in every octave and of their compounds with VIII^{ths}, and of many more such imperfect concords, when the usual flat and sharp sounds are added to complete the scale, has obliged practical musicians, long ago, to distribute that comma, wanting in the fifth *da*, equally among all the four v^{ths}, *CG*, *Gd*, *da*, *aé*, contained in the XVIIth *Cé*. And this interval *Cé* may be increased or decreased a little before it be divided into 4 equal v^{ths}. In any case such distribution is therefore called the Participation or Temperament of the system, and when rightly adjusted is undoubtedly the finest improvement in harmonics.

10. If it be asked why no more primes than 1, 2, 3, 5 are admitted into musical ratios; one reason is, that consonances whose vibrations are in ratios whose terms involve 7, 11, 13, &c, *cæteris paribus* would be less simple and harmonious (*f*) than

(*f*) Sect. III. and Table of the order of the simplicity of consonances.

than thoſe whoſe ratios involve the leſſer primes only.

Another reaſon is this; as perfect fifths and other intervals reſulting from the number 3, make the Schiſm of a comma with the perfect thirds and other intervals reſulting from the number 5, ſo ſuch intervals as reſult from 7, 11, 13, &c, would make other ſchiſms with both thoſe kinds of intervals.

11. The Greek muſicians, after dividing an octave into two 4^{ths}, with the diazeuctic or major tone in the middle between them, and admitting many primes to the compoſition of muſical ratios, ſubdivided the 4th into three intervals of various magnitudes, placed in various orders, by which they diſtinguiſhed their Kinds of Tetrachords (*g*). Two of them have occurred in this Section. The firſt, or $\frac{3}{4}=\frac{8}{9}\times\frac{8}{9}\times\frac{243}{256}$, anſwering to the 4th $=T+T+L$, in Fig. 6, is *Ptolemy*'s *Genus Diatonum ditonicum*, and reſults from that diviſion of a Monochord which bears the name of *Euclid*'s Section of the Canon; the ſecond Kind, or $\frac{3}{4}=\frac{8}{9}\times\frac{9}{10}\times\frac{15}{16}$, anſwering to the 4th $=T+t+H$, in Fig. 3, is *Ptolemy*'s *Diatonum intenſum*.

12. Since the invention of a temperament, all thoſe antient ſyſtems have juſtly been laid aſide, as being unfit for the execution of muſi-

(*g*) Dr. *Wallis* has given a table of them in his Appendix to *Ptolemy*'s Harmonics. Oper. Math. vol. III, pag. 166.

cal compoſitions in ſeveral parts. But to conclude from thence that the antients had no muſic in parts, would be a very weak inference. Becauſe it is much eaſier for practical muſicians to follow the judgment of the ear, which leads naturally to an occaſional temperament of any diſagreeable concords, than to learn and put in practice the theories of philoſophers (*b*): And alſo

(*b*) It may not be amiſs to add the opinion of the famous *Salinas*. Sed unum hoc omnes ſcire volo, inſtrumenta quibus antiqui utebantur, conſonantias habuiſſe imperfectas, ut ea, quibus nunc utimur. Neque enim aliter modulatio convenienter exerceri poterat. Quod ſi de hac conſonantiarum imperfectione, neque *Ptolemæus*, neque alius ex antiquis muſicis mentionem feciſſe reperitur, cauſam potiſſimam eſſe crediderim, quòd ad practicos eam pertinere arbitrarentur; quoniam ſenſu duce ſolùm, non arte aut ratione ſemper fieri ſolita ſit: cujus pleniſſimum et evidentiſſimum teſtimonium reperitur apud *Galenum*, libro primo De Sanitate tuenda, capite quinto; ubi magnam eſſe latitudinem ſanitatis oſtendere volens, ſic inquit: Καὶ τί θαυμαστὸν εἰ τὴν εὐκρασίαν εἰς ἱκανὸν ἐκτείνουσι πλάτος ἅπαντες, ὅπου γε καὶ ἐν αὐταῖς λύραις συαρμοσίαν, τὴν μὲν ἀκριβεστάτην δήπου, μίαν καὶ ἄτμητον ὑπάρχειν εἰκός· ἡ μέν τοι γ' εἰς χρεῖαν ἰοῦσα, πλάτος ἔχει. Πολλάκις γ' οὖν ἡρμόσθαι δοκοῦσαν ἄριστα λύραν, ἕτερος μουσικὸς ἀκριβῶς ἐφηρμόσατο· πανταχοῦ γὰρ ἡ αἴσθησις ἡμῖν ἐστὶ κριτήριον, ὡς πρὸς τὰς ἐν τῷ βίῳ χρείας. hoc eſt, *Quid mirum, ſi Eucraſiam in ſatis amplam latitudinem extendunt univerſi; quando et in lyris conſonantiam ipſam quæ ſumma exactiſſimaque ſit, unicam atque inſectabilem eſſe probabile ſit, et quæ in uſus hominum venit, certe latitudinem habeat. Sæpe namque,* [*quam*] *percommode temperaſſe lyram videaris, alter ſuperveniens muſicus exactius temperavit: ſiquidem nobis ad omnia vitæ munera ſenſus ubique judex eſt.* Ex quibus Galeni verbis liquido conſtat, conſonantias, quibus alſo becauſe we are aſſured from hiſtory, that experience and neceſſity did introduce ſomething of a temperament before the reaſon of it was diſcovered, and the method and meaſure of it was reduced to a regular theory, as in the following propoſition.

SECTION V.

Of the temperaments of imperfect intervals and their ſynchronous variations.

PROPOSITION II.

To reduce the diatonic ſyſtem of perfect conſonances to a tempered ſyſtem of Mean Tones.

Plate III. Fig. 13. When the elements are ranged in this order, T, t, H, T, t, T, H, or this, t, T, H, T, t, T, H, which two we ſhewed to be the beſt (*i*), and the arches *CD*,

 DE,

bus in muſicis utebantur inſtrumentis, jam tunc imperfectas eſſe, quin potius et fuiſſe ſemper et ſemper eſſe futuras. De Muſicâ lib. III. cap. 14. Be it ſo; but did they know, that all the concords cannot be tuned perfect, and why they cannot?

(*i*) Sect. IV. Art. 7.

DE, *EF*, *FG*, *GA*, *AB*, *BC*, are proportional to them, let the major III[d] *CE*, situated between the two hemitones, be bisected in *d*; and let the other two major tones, *FG*, *AB*, be diminished at both ends by the intervals *Ff*, *Gg*, *Aa*, *Bb*, severally equal to half *Dd*; and the octave will then be divided into five mean tones and two limmas, each limma being bigger than the hemitone by a quarter of a comma.

For the interval *Dd* being half the difference between the major and minor tones, *CD*, *DE*, is half a comma (*k*), and therefore the new tone *Cd* or *dE* is an arithmetical mean between them. And each of the temperaments *Ff*, *Gg*, *Aa*, *Bb*, being made equal to half *Dd* or a quarter of a comma, it appears that every major tone is diminished by half a comma, and that every minor tone is as much increased, which reduces all the tones to an equality. And by the construction the limmas *bC*, *Ef* exceed the hemitones by a quarter of a comma apiece. Q. E. D.

Coroll. In the system of mean tones every perfect v[th] is diminished by a quarter of a comma: as will appear by going round the 13[th] figure, and comparing the tempered v[ths], *faC*, *CEg*, *gbd*, *dfa*, *aCE*, *Egb*, with the perfect ones, by means of the notes T, t, H in the angles.

This

(*k*) Sect. II. Art. 4.

This is usually called the vulgar temperament and might be proved several other ways independent of the first and second propositions (*l*).

PRO-

(*l*) *Salinas* tells us, that when he was at Rome, he found the musicians used a temperament there, though they understood not the reason and true measure of it, till he first discovered it, and *Zarlino* published it soon after; first in his Dimonstrationi Harmoniche, Ragionamento quinto, proposta 1ma, and after that, in his Institutioni Harmoniche, part. 2. cap. 43.

After his return into Spain, *Salinas* applied himself to the latin and greek languages, and caused all the antient musicians to be read to him, for he was blind; and in 1577 he published his learned work upon music of all sorts; where treating of three different temperaments of a system, he prefers the diminution of the vth by a quarter of a comma to the other two, which he says are peculiar to certain instruments. De Musicâ Lib. III. cap. 22.

Dechales says, that *Guido Aretinus* was the inventer of that temperament: Ipse nulla habita ratione toni majoris et minoris, hunc unius quintæ defectum aliis omnibus quintis communicat, et quasi dividit, ita ut nulla deficiat nisi quarta parte commatis. Hoc systema, quod valde commodum est, dicitur *Aretini*. Cursus Mathem. Tom. I. pag. 62. De Progressu matheseos et musicæ, cap. 7; et Tom. IV. pag. 15. cap. XI. But that opinion wants confirmation, especially as *Dechales* makes no mention of the claims of *Zarlino* and *Salinas* to that invention; for it seems they had a dispute about it.

PROPOSITION III.

If the five mean tones and the two limmas, that compose a perfect octave, be changed into five other equal tones and two equal limmas, of any indeterminate magnitudes; the synchronous Variations of the limma L, *the mean tone* M, *and of every interval composed of any numbers of them, are all exhibited in the following table, by the numbers and signs of any small indeterminate interval v: And are the same quantities as the variations of the temperaments of the respective imperfect intervals.*

For

2d	IId	3d	IIId	4th	IVth
L	M	L+M	2M	L+2M	3M
$5v$	$-2v$	$3v$	$-4v$	v	$-6v$
$-5v$	$2v$	$-3v$	$4v$	$-v$	$6v$
L+5M	2L+4M	L+4M	2L+3M	L+3M	2L+2M
VIIth	7th	VIth	6th	Vth	5th

For since the perfect VIII$^{th} = 2L + 5M$ is invariable, if the variation of L be put equal to $5v$, as in the table, that of $2L$ is $10v$, and that of $5M$, as being the complement of $2L$ to the VIIIth, is $-10v$; whence the variation of M is $-2v$.

Consequently the variation of the mean 3^{d}, $L+M$, is $5v-2v=3v$, and that of the III^{d}, $2M$, is $-4v$, and that of the mean 4th, $L+2M$, is $5v-4v=v$, and that of the mean IVth, $3M$, is $-6v$.

The variations of the intervals in the lower half of the table, are respectively equal to those in the upper half, but have contrary signs; the corresponding intervals being complements to the perfect octave.

For which reason the compounds of every one of those intervals with any number of octaves, have respectively the same variations both in quantity and quality.

And if the sign of the variation of any one interval be changed, the signs of all the rest will also be changed; because their quantities will vanish all together when v or any one multiple of it vanishes.

As to the second part of the proposition, it will appear in Fig. 13, that any variation v of the mean interval $CdEf$ is the same in quantity as the variation of the temperament Ff of the said interval $CdEf$: and the like is evident in any other instance. Q. E. D.

Coroll. 1. It is obſervable in the table, that the variations of all the major mean intervals II^{d}, III^{d}, IVth, Vth, VIth, VIIth, have the ſame ſign, and thoſe of the minor intervals the contrary ſign.

Coroll. 2. Having extended the circumference *CdEfgabC* of Fig. 13 into a right line, as in Fig. 14, at the points *d*, *E*, *g*, *a*, *b*, that terminate the major mean intervals II^{d}, III^{d}, Vth, VIth, VIIth, meaſured from *C*, (and the minor too meaſured from *c* the other extreme of the octave *Cc*) place the reſpective tabular numbers 2, 4, 1, 3, 5, denoting the proportions of their ſynchronous variations; and in Fig. 15 divide any given line *O* 6 into 6 equal parts, at the points 1, 2, 3, 4, 5; then conceive the 14th Fig. transferred to the 15th five ſeveral times, into five parallel poſitions, ſo that the ſeveral points 1, 2, 3, 4, 5 in each Figure may coincide. And it will be evident, by coroll. 1, that any right line *Ovvvv*, drawn from *O*, terminates the ſynchronous variations, 1*v*, 2*v*, 3*v*, 4*v*, 5*v*, of thoſe mean intervals, Vth, II^{d}, VIth, III^{d}, VIIth, the variations being meaſured from their reſpective origins 1, 2, 3, 4, 5; and that theſe are alſo the ſynchronous variations of the temperaments of the reſpective imperfect intervals, and of their complements to the VIIIth and compounds with VIII^{ths}, that is, of all the intervals in the ſyſtem.

For as to the mean IVth *fb*, Fig. 14, its contemporary variation in Fig. 15, will be the line

6*v*

$6v$ in the ſixth parallel $F6B'F'$, when its temperament $B'6$ or $B'b'$ is taken equal to twice Bb and placed the ſame way from its origin b' or 6. Becauſe in Fig. 14 the temperament of the imperfect IV^th^ fb is $Ff+Bb=2Bb$.

As want of room in Fig. 15 will not permit the ſeveral intervals CG, CD, &c. even leſs than one octave, to be repreſented in their due proportions to $G1$, the quarter of the comma, which is but the 223^d^ part of an octave; we muſt conceive them continued far beyond the margin of the paper.

Coroll. 3. When the III^d^ is perfect, the temperaments belonging to the V^th^ and VI^th^ are ſeverally $\frac{1}{4}$ of a comma, the former in defect, the latter in exceſs: and if either of them be made leſs, the other will be greater than $\frac{1}{4}$ comma.

Pl. III. & IV. Fig. 15, 16. For when the Temperer $Ovvv$ falls upon E, the III^d^ CE is perfect, and the tempered V^th^ $C1$ is leſs than the perfect V^th^ CG by $G1$, and the tempered VI^th^ $C3$ is bigger than the perfect VI^th^ CA by $A3=G1=\frac{1}{4}$ comma.

Hence when Av, any other temperament belonging to the VI^th^, is leſs than $A3$ or $\frac{1}{4}$ comma, Gv the correſponding temperament belonging to the V^th^, is greater than $G1$ or $\frac{1}{4}$ comma: and on the contrary, when Gv is leſs than $G1$, the reſpective Av is bigger than $A3$. And whatever be the magnitudes of theſe temperaments of the V^th^ and VI^th^, thoſe of their com-

complements to the VIIIth and compounds with VIII^{ths} are the ſame.

Coroll. 4. When the VIth is perfect, the temperaments of the Vth and III^{d} are ſeverally $\frac{1}{3}$ comma, and are both negative.

Fig. 16. For when the temperer $Ovvv$ falls upon the line $OHAI$, the temperament of the VIth vaniſhes, and thoſe of the Vth and III^{d} are GH and EI, and are equal. For the equal lines $G1$, $A3$ and equal triangles GEO, AOE ſhew, that the line GE is parallel to AO; whence GH is equal to EI, and the ſimilar triangles IEO, $A3O$ give $IE = \frac{4}{3}A3 = \frac{4}{3} \times \frac{1}{4}$comma$= \frac{1}{3}$ comma.

Coroll. 5. When the Vth is perfect, the temperaments of the VIth and III^{d} are ſeverally equal to a comma in exceſs.

For when the temperer $Ovvv$ falls upon the line $OGKL$, the temperament of the Vth vaniſhes, and thoſe of the VIth and III^{d} are now AK and EL, which are equal, becauſe of the parallelograms $AEGO$, $AELK$; and EL is $= 4 \times G1$ or four quarters of a comma.

Coroll. 6. When the temperer $Ovvv$ falls within the angle AOE, the temp^{t}. Vth = temp^{t}. VIth + temp^{t}. III^{d}, that is, the line $Gv = Av + Ev$, or the lines $G1 + 1v = A3 - 3v + Ev$, that is, putting the letter v for the line $1v$, $\frac{1}{4}c + v = \frac{1}{4}c - 3v + 4v$, which is evidently true.

Coroll. 7. When the temperer $Ovvv$ falls within the angle EOG, the temp^{t}. VIth = temp^{t}. Vth + temp^{t}. III^{d}, that is, the line $Av = Gv + Ev$,

Ev, or the line $sA3+3v=G1-1v+Ev$, that is, putting v for the line $1v$, $\frac{1}{4}c+3v=\frac{1}{4}c-v+4v$, which is true.

Coroll. 8. When the temperer falls any where out of the angle AOG, the temp^t. III^d=temp^t. vth+temp^t. VIth, that is, when it falls beyond the ſide AO, the temp^t. $EI+Iv=GH+Hv+Av$, or putting the letter v for the line Hv, $\frac{1}{3}c+4v=\frac{1}{3}c+v+3v$, which is true: and when the temperer falls beyond the other ſide OG, the ſaid temp^t. $EL+Lv=Gv+AK+Kv$, that is, putting v for the line Gv, $c+4v=v+c+3v$, which is true.

Coroll. 9. The ſum of the temperaments of the vth and VIth is $\frac{1}{2}$ a comma when the III^d is perfect; is leſs than $\frac{1}{2}$ a comma by $\frac{1}{2}$ the temperament of the III^d when flattened; and greater than $\frac{1}{2}$ a comma by $\frac{1}{2}$ the temperament of the III^d when ſharpened.

For in the firſt caſe the ſaid ſum is $G1+A3$; in the ſecond, it is $G1+1v+A3-3v=G1+A3-2v$; and in the third, it is $G1-1v+A3+3v=G1+A3+2v$; in which latter caſes the temperament of the III^d is $4v$.

Coroll. 10. Hence the ſum of the temperaments of all the concords is leſs when the III^{ds} are flattened, than the like ſum when the III^{ds} are equally ſharpened; and the ſum is the leaſt of all when the III^{ds} are perfect, as in the ſyſtem of mean tones (*m*).

(*m*) Prop. II.

Scholium.

Scholium.

From the third and tenth corollaries I think we might juſtly pronounce the ſyſtem of mean tones to be the beſt poſſible, were it evident that equal temperaments cauſe different concords to be equally diſagreeable to the ear (*n*).

But if it ſhall appear, that the VIth and 3^{d} and their compounds with octaves, are more diſagreeable in their kind, than the Vth and 4th and their compounds with octaves, all being equally tempered, as in that ſyſtem; will it not follow, that the temperament of the former Parcel of concords ſhould be ſmaller than that of the latter, to make them all as equally harmonious as poſſible, without ſpoiling the harmony of the III^{d} and 6th and their compounds with octaves; which third parcel makes up the ſum of all the concords in the ſyſtem.

For

(*n*) Mr. *Huygens* has pronounced it the beſt, in ſaying that the muſicians in the other planets may know perhaps, cur optimum ſit temperamentum in chordarum ſyſtemate, cum ex diapente quarta pars commatis ubique deciditur; Coſmotheoros pag. 76; but has given us no reaſon for his aſſertion, either in that incomparable book or in his Harmonic Cycle; where he only appeals to the approbation and practice of muſicians and refers to the demonſtrations of *Zarlino* and *Salinas*. But neither of theſe celebrated authors do any thing more, if I rightly remember, (for I have not the books now by me) than reduce the Diatonic ſyſtem of perfect conſonances to that of mean tones, by diſtributing the ſchiſm of a whole comma into quarters; not at all conſidering, whether thoſe equal temperaments have the ſame, or a different effect upon the ſeveral concords.

For if it be the immediate ſucceſſion of a worſe harmony to a better, as in inſtruments badly tuned, which chiefly offends the ear; it muſt be allowed, that a ſyſtem would be the better, *cæteris paribus*, for having all the concords as equally harmonious in their kinds, as the nature and properties of numbers will permit.

In order to reſolve thoſe queſtions upon philoſophical principles, and to determine the temperament of a given ſyſtem, that ſhall cauſe all the concords, at a medium of one with another, to be equally, and the moſt harmonious in their ſeveral kinds, I found it neceſſary to make a thorough ſearch into the abſtract nature and properties of tempered conſonances; and thence to derive their effects upon our organs of hearing: A large field of harmonics hitherto uncultivated.

But before I enter upon it, it will be convenient to finiſh this ſection with a determination of the leaſt ſum of any three temperaments in different parcels, when any two of them have any given ratio.

PRO-

PROPOSITION IV.

To find a ſet of temperaments of the Vth, VIth *and* III^{d} *upon theſe conditions; that thoſe of the* Vth *and* VIth *ſhall have the given ratio of* r *to* s, *and the ſum of all three ſhall be the leaſt poſſible.*

Pl. V. VI. Part of the 17th and 18th Figures being conſtructed like the 15th, from A towards K take $AM : G1 :: s : r$, and through the interſection p of the lines AG, $M1$, draw the temperer $Orst$; I ſay Gr, As, Et are the temperaments required.

For by the ſimilar triangles Grp, Asp, and $G1p$, AMp, we have $Gr : As ::$ ($Gp : Ap :: G1 : AM ::$) $r : s$ by conſtruction, as required by the firſt condition.

Again, in the ſame line MAC take $AN = AM$, and through the interſection P of the lines AG, $N1$ produced, draw another temperer $ORST$; and by the ſimilar triangles GRP, ASP, and $G1P$, ANP, we have $GR : AS ::$ ($GP : AP :: G1 : AN$ or $AM ::$) $r : s$ by conſtruction, which likewiſe anſwers the firſt condition; and it is eaſy to underſtand, that no other temperers but thoſe two can anſwer that condition.

Now

Now whatever be the quantity and quality of the given ratio r to s, I ſay the ſum $Gr+As+Et$ is leſs than $GR+AS+ET$.

Caſe 1. Fig. 17. For when r is bigger than s, or the ratio of r to s, or of $G1$ or $A3$ to AM or AN, is a ratio of majority, the temperers Op, OP fall within the angles AOE, AOC reſpectively; as appears by the conſtruction. Whence, by coroll. 6 and 8. prop. III, $Gr=As+Et$, and $ET=GR+AS$; and therefore $Gr+As+Et : GR+AS+ET :: Gr : ET$, which is a ratio of minority, becauſe Gr is leſs than GH or EI (*o*) and EI leſs than ET.

Caſe 2. Fig. 18. When r is leſs than s, or the ratio of r to s, or of $G1$ to AM or AN is a ratio of minority, the temperers Op, OP fall within the angles EOG, AOC reſpectively; as appears by the conſtruction. Whence, by coroll. 7 and 8. prop. III, $As=Gr+Et$, and $ET=GR+AS$, and therefore $Gr+AS+Et : GR+AS+ET :: As : ET$, which is a ratio of minority; becauſe $Gr : As :: r : s :: GR : AS$, whence, as Gr is leſs than GR, ſo As is leſs than AS, which is leſs than IT, which is leſs than ET.

Caſe 3. Fig. 17 and 18. When r to s, or $G1$ to AM or AN, is the ratio of equality, the temperer $Orst$ coincides with the line OE, and $Orst$ is parallel to GA; whence it is plain, that the ſum of the temperaments $G1+A3+o$, is leſs than $GR+AS+ET$, as required. Q. E. D.

Coroll.

(*o*) See coroll. 4. Prop. III.

Coroll. Putting c for the comma EL or four G 1, when the temp^t. V : temp^t. VI : : $r : s$, the required temperaments of the V, VI and III are, $Gr = \frac{r}{3r+s} c$, $As = \frac{s}{3r+s} c$ and $\pm Et = \frac{r-s}{3r+s} c$. And according as r is bigger or leſs than s, the temperer $Orst$ falls within the angle AOE or EOG.

Fig. 17 and 18. For, $As : Gr :: s : r$, and $As : 3Gr$ or $sK :: s : 3r$, and $As : As + sK$ or c (p) $:: s : s + 3r$. Whence $As = \frac{s}{3r+s} c$, and $Gr = \frac{r}{s} As = \frac{r}{3r+s} c$, and in the angle AOE, $Et = Gr - As = \frac{r-s}{3r+s} c$, but in EOG, $Et = As - Gr$, by the equations in caſe 1, 2.

PROPOSITION V.

To find a ſet of temperaments of the Vth, VIth *and* III^{d} *upon theſe conditions; that thoſe of the* Vth *and* III^{d} *ſhall have the given ratio of* r *to* t, *and the ſum of all three ſhall be the leaſt poſſible.*

Pl. VII. VIII. Fig. 19, 20. If t to r be a ratio of minority, or of equality, or even of majority leſs than 1 to $\frac{1+\sqrt{33}}{8}$ or 0.843070 &c, from E towards I take $EM : G$ 1 $:: t : r$, and through

(p) See Dem. coroll. 5. prop. III.

through the interſection p of the lines $M1$, GE produced, draw the temperer $Orst$, and the required temperaments will be Gr, As, Et.

But if the ratio of t to r be greater than 1 to 0. 843070 &c, in Fig. 20, from E towards L take $EN : G1 :: t : r$, and through the interſection P of the lines $N1$, GE, draw the temperer $ORST$, and the required temperaments will be GR, AS, ET.

And if $t : r :: 1 : 0.843070$ &c, the required temperaments will be Gr, As, Et, or GR, AS, ET, their ſums being equal.

In the firſt caſe, Fig. 19, take $EN = EM$, and in the ſecond, Fig. 20, $EM = EN$; and through the interſections P, p of the lines $N1$, $M1$ with GE, draw two more temperers $ORST$, $Orst$.

Then by the ſimilar triangles Grp, Etp and $G1p$, EMp, we have $Gr : Et :: (Gp : Ep :: G1 : EM ::) \; r : t$ by conſtruction, as required by the firſt condition.

Again, by the ſimilar triangles GRP, ETP and $G1P$, ENP, we have $GR : ET :: (GP : EP :: G1 : EN ::) \; r : t$ by conſtruction, which alſo anſwers the firſt condition; and it is eaſy to underſtand that no other temperers but thoſe can anſwer that condition.

Caſe 1. Fig. 19. Now when t is to r, and therefore EM or EN to $G1$ in a ratio of minority, the temperers Op, OP fall within the angles AOE, EOG reſpectively by the con-

 ſtruction.

struction. Whence, by coroll. 6 and 7. prop. 3, $Gr = As + Et$ and $AS = GR + ET$.

But $Gr : Et :: r : t$, and $Gr : \frac{1}{4} Et$ or $r1 :: r : \frac{1}{4} t$, and $Gr : Gr - r1$ or $\frac{1}{4} c :: r : r - \frac{1}{4} t :: 4r : 4r - t$. Whence $Gr = \frac{r}{4r-t} c$, and $Et = \frac{t}{r} Gr = \frac{t}{4r-t} c$, and $As = Gr - Et = \frac{r-t}{4r-t} c$, by the equation in the last paragraph.

Likewise $Gr : ET :: r : t$, and $GR : \frac{1}{4} ET$ or $R1 :: r : \frac{1}{4} t$, and $GR : GR + R1$ or $\frac{1}{4} c :: r : r + \frac{1}{4} t :: 4r : 4r + t$. Whence $GR = \frac{r}{4r+t} c$, and $Et = \frac{t}{r} GR = \frac{t}{4r+t} c$, and $AS = GR + ET = \frac{r+t}{4r+t} c$, by the equation above.

Therefore $Gr + As + Et : GR + AS + ET :: Gr : AS :: \frac{r}{4r-t} : \frac{r+t}{4r+t} :: 4rr + rt : 4rr + rt, + 2rt - tt$, which is a ratio of minority; because t being less than r, tt is less than $2rt$.

Case 2. Fig. 20. When t to r, and therefore EM or EN to $G1$, is a ratio of majority, the temperers Opt, OPT fall within the angles, AOC, EOG respectively; as appears by the construction. Whence, by coroll. 8 and 7 prop. 3, $Et = Gr + As$ and $AS = GR + ET$.

In which case the theorems for the values of Gr, As, Et, GR, AS, ET are the same as before.

Therefore $Gr + As + Et : GR + AS + ET :: Et : AS :: \frac{t}{4r-t} : \frac{r+t}{4r+t} :: 4rt + tt : (4rr +$

4rt

$4rt - rt - tt$ or) $4rt + tt$, $+4rr - rt - 2tt$, which is a ratio of minority, except either when $4rr - rt - 2tt = 0$, or $\frac{4rr - rt - 2tt}{4tt} = 0$, or $\frac{rr}{tt} - \frac{r}{4t} - \frac{1}{2} = 0$, which gives $\frac{r}{t} = \frac{1 + \sqrt{33}}{8} = 0.843070$ &c (q); or when $4rr - rt - 2tt$, or $\frac{rr}{tt} - \frac{r}{4t} - \frac{1}{2}$ is negative, and consequently $\frac{r}{t}$ is less than 0. 843070 &c (r), or the ratio of t to r is greater than 1 to 0. 843070 &c.

In the first case either $Gr + As + Et$ or $GR + AS + ET$, as being equal, are the required temperaments; in the second the latter only, as being less than the former.

Case 3. When $t = r$, we have EM or $EN = G1$; therefore the intersection p is removed to an infinite distance, and the temperer $Orst$ coincides

(q) For supposing $\frac{r}{t} = \frac{x}{1} = x$, we have $\frac{rr}{tt} - \frac{r}{4t} - \frac{1}{2} = xx - \frac{1}{4}x - \frac{1}{2} = 0$. Whence $xx - \frac{1}{4}x = \frac{1}{2}$ and $xx - \frac{1}{4}x + \frac{1}{8} \times \frac{1}{8} = \frac{1}{8} \times \frac{1}{8} + \frac{1}{2} = \frac{1}{8} \times \frac{1}{8} + \frac{32}{8 \times 8} = \frac{33}{8 \times 8}$ whose square roots are $x - \frac{1}{8} = \frac{\pm\sqrt{33}}{8}$; whence x or $\frac{r}{t} = \frac{1 \pm \sqrt{33}}{8} = \frac{1 \pm 5.744562 \text{ \&c.}}{8}$

(r) For since the root 0. 843070 &c, when substituted for x, will make the value of $xx - \frac{1}{4}x - \frac{1}{2}$, or of $x - \frac{1}{4} - \frac{1}{2x} = 0$; a smaller number substituted for x, will produce a negative value of the latter, and consequently of the former quantity.

cides with $OHAI$. Hence $Gr+As+Et$ becomes $=GH+0+EI$, and is to $GR+AS+ET$ :: 5 : 6, a ratio of minority, produced by putting $t=r$ in the terms of that ratio in case 1 or 2. Q. E. D.

Coroll. When the temp^{t}. v : temp^{t}. III :: r : t, if $\frac{r}{t}$ be bigger than 0.843070 &c, the required temperaments of the v, VI and III are, $Gr=\frac{r}{4r-t}c$, $As=\frac{r-t}{4r-t}c$, $Et=\frac{t}{4r-t}c$. And the temperer $Orst$ falls within the angle AOE or AOC, according as r is bigger or less than t.

But if $\frac{r}{t}$ be less than 0.843070 &c, they are $GR=\frac{r}{4r+t}c$, $AS=\frac{r+t}{4r+t}c$, $ET=\frac{-t}{4r+t}$; and the temperer $ORST$ falls within the angle EOG.

And if $\frac{r}{t}=0.843070$ &c, their sums are equal and either of them answers the problem.

PRO-

PROPOSITION VI.

To find a ſet of temperaments of the v^th^, vi^th^ *and* iii^d^ *upon theſe conditions that thoſe of the* vi^th^ *and* iii^d^ *ſhall have the given ratio of* s *to* t, *and the ſum of all three ſhall be the leaſt poſſible.*

Pl. IX. X. Fig. 21 and 22. From E towards C take $EM : A3 :: t : s$ and through the interſection p of the lines $M3$, AE draw the temperer $Orst$, and the required temperaments will be Gr, As, Et.

For by the ſimilar triangles Asp, Etp and $A3p$, EMp, we have $As : Et :: (Ap : Ep :: A3 : EM ::)\ s : t$ by conſtruction, as required by the firſt condition.

Again, taking $EN = EM$, through the interſection P of the lines $N3$, AE produced, draw the temperer $ORST$, and by the ſimilar triangles ASP, ETP and $A3P$, ENP, we have $AS : ET :: (AP : EP :: A3 : EN$ or $EM) :: s : t$ by conſtruction, which alſo anſwers the firſt condition; and it is plain that thoſe are all the temperers which can anſwer it.

Now whatever be the ratio of s to t, I ſay that $Gr + As + Et$ is leſs than $GR + AS + ET$.

Case 1. Fig. 21. When t is to s, or EM to $A3$ in a ratio of minority, the temperers Op, OP fall within the angles AOE, EOG respectively, as appears by the construction. Whence by coroll. 6 and 7 prop. III, $Gr=As+Et$ and $AS=GR+ET$.

But $Et : As :: t : s$, and $Et : \frac{4}{3}As$ or $It :: t : \frac{4}{3}s$ and $Et : Et+It$ or $\frac{1}{3}c$ (s) $:: t : t+\frac{4}{3}s :: 3t : 3t+4s$. Whence $ET=\frac{t}{4s+3t}c$. And $As=Et\times\frac{s}{t}=\frac{s}{4s+3t}c$. And $Gr=As+Et=\frac{s+t}{4s+3t}c$, by the equation in the last paragraph.

Again, $ET : AS :: t : s$ and $ET : \frac{4}{3}AS$ or $IT :: t : \frac{4}{3}s$ and $ET : IT-ET$, or IE or $\frac{1}{3}c :: t : \frac{4}{3}s-t :: 3t : 4s-3t$. Whence $ET=\frac{t}{4s-3t}c$ and $AS=ET\times\frac{s}{t}=\frac{s}{4s-3t}c$.

Therefore $Gr+As+Et : GR+AS+ET :: Gr : AS :: \frac{s+t}{4s+3t} : \frac{4s-3t}{s} :: 4ss+4st-3st-3tt$, or $4ss+3st-2st-3tt : 4ss+3st$, which is evidently a ratio of minority.

Case 2. Fig. 22. When t is to s, or EM to $A3$ in a ratio of majority, the temperers Op, OP fall within the angles AOE, GOc respectively. Whence, by coroll. 6 and 8 prop. III, $Gr=As+Et$ and $ET=GR+AS$, and $Gr+As+Et : GR+AS+ET :: Gr : ET$, which is plainly a ratio of minority.

Case

(s) See Dem. coroll. 4. prop. III.

Case 3. When $t=s$ or EM or $EN=A3$, the interſection P vaniſhes, and the temperer $ORST$ coincides with $OGKL$, as appears by the conſtruction. Whence by the concluſion of the ſecond caſe, $Gr+As+Et : o+AK+EL :: Gr : EL$, a ratio of minority, as before. Q. E. D.

Coroll. When the temp^t. VI : temp^t. III :: $s : t$, the required temperaments of the V, VI and III are, $Gr=\frac{s+t}{4s+3t}c$, $As=\frac{s}{4s+3t}c$, $Et=\frac{t}{4s+3t}c$; and the temperer lies within the angle AOE, whatever be the quantity and quality of the ratio of s to t.

Scholium.

Theſe three problems comprehend the ſolution of a more general one, namely, To find the temperament of a ſyſtem of ſounds upon theſe conditions; that the octaves be perfect, that the ratio of the temperaments of any two given concords in different parcels be given, and that the ſum of the temperaments of all the concords, be the leaſt poſſible.

The reaſon is, that the given ratio of the temperaments of any two concords, determines the poſition of the temperer of the ſyſtem, and this the three magnitudes of the temperaments of all the concords, whatever be their number. But if both the given concords be contained in any one of the three parcels above mentioned (*t*), the given ratio of their temperaments can be no other than that of equality; and this *datum* is plainly insufficient.

SECTION VI.

Of the Periods, Beats and Harmony of imperfect consonances.

DEFINITIONS.

I. Any two sounds whose single vibrations have any small given ratio, are called Imperfect Unisons:

II. And the cycle of their pulses is called Simple or Complex, according as the difference of the least terms of that ratio is an unit or units:

III. And when a complex cycle is divided into as many equal parts as that difference contains units, each part is called a Period of the pulses:

IV. And the cycles of perfect consonances are often called Short cycles, to distinguish them from the long cycles of imperfect unisons.

PRO-

(*t*) Schol. prop. III.

PROPOSITION VII.

In going from either end to the middle of any ſimple cycle or period of the pulſes of imperfect uniſons, the Alternate Leſſer Intervals between the ſucceſſive pulſes increaſe uniformly, and are proportional to their diſtances from that end; and at any diſtances from it leſs than half the ſimple cycle or period, are leſs than half the leſſer of the two vibrations of the imperfect uniſons.

Let the vibrations be V and v, and V : v : : R : r, the integers R, r being the leaſt in that ratio; and putting $d=R-r$, we have the complex cycle $r\mathrm{V}=Rv=rv+dv$ (*u*), and the period $\frac{r}{d}\mathrm{V}=\frac{r}{d}v+v$, which when $d=1$, is a ſimple cycle (*x*).

Pl. XI. Fig. 23, 24, 25, 26. To aſſiſt the imagination, let the ſucceſſive vibrations V, V, V, &c, be repreſented by the equal lines *AB*, *BC*, *CD*, &c, and the middle inſtants of their pulſes

(*u*) Sect. III. Art. 12. (*x*) Def. 11.

pulses (y) by the points *A*, *B*, *C*, &c; and the successive vibrations v, v, v, &c. by the equal lines ab, bc, cd, &c, and the middle instants of their pulses by the points *a*, *b*, *c*, &c.

Then beginning from two coincident pulses at *A* or *a*, it is observable, that the successive intervals of the pulses are alternately bigger and less; and that the Alternate Lesser Intervals Bb, Cc, Dd, &c, or $V-v$, $2V-2v$, $3V-3v$, &c, increase uniformly, by the repeated addition of the first lesser interval $V-v$, at every equal increment V or v of their distances from *A*. The alternate lesser intervals are therefore proportional to their distances from the coincident pulses *A*, *a*.

Now any assigned distance $3V : rV :: 3v : rv :: 3V-3v : rV-rv=dv$, by the equation; whence $3V : \frac{r}{d}V :: 3V-3v : v$; consequently if the assigned distance 3V or AD be less than half the simple cycle or period $\frac{r}{d}$ V, the adjoining interval $3V-3v$, or Dd is less than half v; but if bigger, bigger than half v.

And the argument is the same in going backwards from the two next coincident pulses at U and w, U and x, &c, their larger and lesser alternate intervals being evidently of the same magnitudes as in going forwards.

Fig. 24. Now if the difference $d=2$, and the length of the complex cycle be the line AU or $ax=rV=rv+2v$, having divided it into two equal

(y) Sect. 1. Art. 12.

equal parts AX, XU, we have the part or period $AX = \frac{r}{2}V$; which becauſe 2 does not meaſure r (z), conſiſts of a multiple of V, as AK, and a remainder $KX = \frac{1}{2}V = \frac{1}{2}KL$.

We have alſo, by the ſame equation, $AX = \frac{r}{2}v + v$, which becauſe 2 does not meaſure r, conſiſts of a multiple of v (one more than that other of V) as Al, and a like remainder $lX = \frac{1}{2}v = \frac{1}{2}lm$.

Now the diſtances of the ſucceſſive pulſes of V from the point X are XL, XM, XN, &c, or $\frac{1}{2}V$, $\frac{3}{2}V$, $\frac{5}{2}V$, &c, and thoſe of the ſucceſſive pulſes of v are Xm, Xn, Xo, &c, or $\frac{1}{2}v$, $\frac{3}{2}v$, $\frac{5}{2}v$, &c, and the differences of thoſe reſpective diſtances, or the alternate leſſer intervals between the ſucceſſive pulſes of V and v, are Lm, Mn, No, &c, or $\frac{1}{2}V - \frac{1}{2}v$, $\frac{3}{2}V - \frac{3}{2}v$, $\frac{5}{2}V - \frac{5}{2}v$, &c; which increaſe uniformly by the repeated addition of $V - v$ or $\frac{2}{2V - 2v}$ to the firſt and ſucceeding intervals.

Aſſign any diſtances XL, XN, or $\frac{1}{2}V$ and $\frac{5}{2}V$; then $\frac{1}{2}V : \frac{5}{2}V :: \frac{1}{2}v : \frac{5}{2}v :: \frac{1}{2}V - \frac{1}{2}v : \frac{5}{2}V - \frac{5}{2}v$, that is, $XL : XN :: Xm : Xo :: Lm : No$, or the alternate leſſer intervals are proportional to their diſtances from the periodical point X.

Now

(z) For if 2 meaſured r, it would alſo meaſure $R = r + 2$, and ſo the terms R, r of the ratio V to v would not be the leaſt, as they are ſuppoſed to be.

Now any aſſigned diſtance $\frac{5}{2}V : \frac{r}{2} V :: \frac{5}{2}v : \frac{r}{2} v :: \frac{5}{2}V - \frac{5}{2}v : \frac{r}{2} V - \frac{r}{2} v = v$, by the given equation, that is, $\frac{5}{2}V : \frac{r}{2} V :: \frac{5}{2}V - \frac{5}{2}v : v$; conſequently if the aſſigned diſtance $\frac{5}{2}V$ or XN, be leſs than half the period $\frac{r}{2} V$ or half XU, the adjoining interval $\frac{5}{2}V - \frac{5}{2}v$ or No, is leſs than half v; but if bigger, bigger than half v.

And in going backwards from X, the alternate leſſer intervals Kl, Ik, Hi, &c, are reſpectively equal to Lm, Mn, No, &c, at equal diſtances on each ſide of X.

Fig. 25. In like manner if $d=3$, or $rV = rv + 3v$, having divided this cycle AU or ay into three equal periods AX, XY, YU, that equation gives $AX = \frac{r}{3} V$, which conſiſts of a multiple of V, as AG, and a remainder $GX = \frac{1}{3}V$ (or $\frac{2}{3}V$ hereafter to be conſidered) $= \frac{1}{3} GH$, whoſe complement $XH = \frac{2}{3}V$.

The ſame period AX is alſo $= \frac{r}{3} v + v$ by the ſame equation, and therefore conſiſts of a multiple of v (one more than that other of V) as Ah, and a like remainder $hX = \frac{1}{3}v = \frac{1}{3}hi$, whoſe complement $Xi = \frac{2}{3}v$.

Hence the diſtances from X of the ſucceſſive pulſes of V are XH, XI, XK, &c, or $\frac{2}{3}V$, $\frac{5}{3}V$, $\frac{8}{3}V$, &c, and thoſe of the ſucceſſive pulſes of v are Xi, Xk, Xl, &c, or $\frac{2}{3}v$, $\frac{5}{3}v$, $\frac{8}{3}v$, &c. and their differences, or the alternate

leſſer

leſſer intervals between the ſucceſſive pulſes of V and v, are Hi, Ik, Kl, &c, or $\frac{2}{3}V-\frac{2}{3}v$, $\frac{5}{3}V-\frac{5}{3}v$, $\frac{8}{3}V-\frac{8}{3}v$, &c; which increaſe uniformly by the repeated addition of $V-v$ or $\frac{3V-3v}{3}$ to the firſt and ſucceeding intervals.

Aſſign any diſtances XH and XK, or $\frac{2}{3}V$ and $\frac{8}{3}V$; then $\frac{2}{3}V : \frac{8}{3}V :: \frac{2}{3}v : \frac{8}{3}v, :: \frac{2}{3}V-\frac{2}{3}v : \frac{8}{3}V-\frac{8}{3}v$, that is, $XH : XK :: Xi : Xl :: Hi : Kl$, or the alternate leſſer intervals are proportional to their diſtances from the periodical point X.

Now any aſſigned diſtance $\frac{8}{3}V : \frac{r}{3}V :: \frac{8}{3}v : \frac{r}{3}v :: \frac{8}{3}V-\frac{8}{3}v : \frac{r}{3}V-\frac{r}{3}v=v$ by the equation, that is, $\frac{8}{3}V : \frac{r}{3}V :: \frac{8}{3}V-\frac{8}{3}v : v$; ſo that if the aſſigned diſtance $\frac{8}{3}V$ or XK be leſs than half the period $\frac{r}{3}V$ or half XY, the adjoining interval $\frac{8}{3}V-\frac{8}{3}v$ or Kl is leſs than half v; but if bigger, bigger than half v.

By doubling the period $AX=AG+\frac{1}{3}V$, we have $AY=2AG+\frac{2}{3}V=AN+\frac{2}{3}V$, ſo that NY is$=\frac{2}{3}V$ and its complement $YO=\frac{1}{3}V$. Again by doubling $AX=Ab+\frac{1}{3}v$, we have $AY=2Ab+\frac{2}{3}v=Ap+\frac{2}{3}v$, ſo that $pY=\frac{2}{3}v$ and its complement $Yq=\frac{1}{3}v$.

Hence the alternate leſſer intervals of the pulſes of V and v, in going oppoſite ways to equal diſtances from X and from Y, are equal. And in going contrary ways from X towards A, and from Y towards U, the alternate leſſer intervals

intervals are $\frac{1}{3}V - \frac{1}{3}v$, $\frac{4}{3}V - v\frac{4}{3}$, $\frac{7}{3}V - \frac{7}{3}v$, &c, which increafe uniformly as before; and $\frac{7}{3}V$ being an affigned diftance from *X* or *Y*, we have $\frac{7}{3}V : \frac{r}{3}V :: \frac{7}{3}v : \frac{r}{3}v :: \frac{7}{3}V - \frac{7}{3}v : \frac{r}{3}V - \frac{r}{3}v = v$ as before. So that if the affigned diftance $\frac{7}{3}V$ be lefs than half the period $\frac{r}{3}V$, the adjoining interval $\frac{7}{3}V - \frac{7}{3}v$ is lefs than half v; but if bigger, bigger than half v.

Fig. 26. Laftly when the period $AX, = \frac{r}{3}V$, confifts of a multiple of V as AG and a remainder $GX = \frac{2}{3}V$, which remained to be confidered, its complement XH is $= \frac{1}{3}V$, and the demonftration would proceed in the fame method as before.

Whoever defires a general proof of the propofition for any value of the difference *d*, need only read the laft example over again with a defign to make the proof general; and he will perceive that what has been faid of the number 3 as a value of *d*, *mutatis mutandis*, is plainly applicable to any other value. Q. E. D.

Coroll. 1. Any fimple cycle or period of the pulfes of imperfect unifons contains one more of the quicker than of the flower vibrations, as appears by its equation, $\frac{r}{d}V = \frac{r}{d}v + v$; and the periodical points *X*, *Y*, &c, always fall within thofe values of v that are feverally contained within as many correfponding values of V, and the number of thofe points in each complex cycle is $d - 1$.

Coroll.

Coroll. 2. The leffer intervals that lie neareft to the periodical points and the points of coincidence, are lefs than any of the reft and are $\frac{V-v}{d}$ and all its multiples, whereof the greateft multiplier is d; as $\frac{V-v}{3}$, $\frac{2V-2v}{3}$, $\frac{3V-3v}{3}$, when $d=3$; $\frac{V-v}{4}$, $\frac{2V-2v}{4}$, $\frac{3V-3v}{4}$, $\frac{4V-4v}{4}$, when $d=4$; &c.

Coroll. 3. Some of the alternate leffer intervals of the pulfes of imperfect unifons, are the differences of equal numbers of their vibrations, counted from the neareft coincident pulfes; and others are the differences of equal numbers of the fame part or parts of their fingle vibrations, counted from the neareft periodical point.

Coroll. 4. If the vibrations of two couples of imperfect unifons, or of any two confonances, be proportional, the periods and cycles of their pulfes, whether fimple or complex, will be in the ratio of the homologous vibrations.

Let T and t be the vibrations of one couple, and V and v thofe of the other; and fince $T : t :: V : v :: r+d : r$, the cycles of their pulfes are $rT = \overline{r+d} \times t$ and $rV = \overline{r+d} \times v$, and the periods are $\frac{r}{d} T = \frac{r+d}{d} t$ and $\frac{r}{d} V = \frac{r+d}{d} v$; and are in the ratio of T to V, or of t to v.

Coroll. 5. The length of the period of the Leaft Imperfections in any confonance of imperfect unifons, is the fame as that of the period of its pulfes.

Pl. XI.

Pl. XI. Fig. 23, 24, 25, 26. For unisons are perfect when their successive pulses are constantly coincident (*a*), and imperfect when the ratio of their vibrations is a little altered from the ratio of equality (*b*); and then the pulses are gradually separated by Alternated Lesser Intervals, which are the Imperfections of this consonance; and since they increase in going from the beginning to the middle of every simple cycle, or period of the pulses, and thence decrease to the end of it (*c*), the length of the period of the Least Imperfections of imperfect unisons is plainly the same as that of the period of their pulses.

PROPOSITION VIII.

If either of the vibrations of imperfect unisons and any multiple of the other, or any different multiples of both, whose ratio is irreducible, be considered as the single vibrations of an imperfect consonance, the length of the period of its least imperfections, will be the same as that of the pulses of the imperfect unisons.

Pl. XI. Fig. 23, 27. For instance, if AB and ab be the vibrations of imperfect unisons, $2AB$

(*a*) Sect. I. Art. 3. (*b*) Sect. VI. Defin. I.
(*c*) Prop. VII.

$2AB$ or AC and ab will be the vibrations of imperfect octaves, whose treble is one of the unisons, and whose base is derived from the other by intermitting every other pulse of the series, A, B, C, D, E, &c.

Now if these octaves were perfect, every pulse of the base would coincide with every other pulse of the treble; but here they are gradually separated by some of the alternate lesser intervals Cc, Ee, &c, of the imperfect unisons. The intermediate pulses of the treble, which in perfect octaves would bisect the intervals of the pulses of the base, are also gradually separated from the round points which bisect them, by the rest of the alternate lesser intervals of the said unisons. And thus the imperfections of the tempered octaves, or the Dislocations of the pulses in their successive short cycles (*d*), are every where the same as the imperfections of the unisons, and consequently have the same periods.

The argument is the same if $2ab$ or ac and AB be the vibrations of imperfect octaves, as in Fig. 28; and also if any other multiple of AB or ab, as mAB or mab, be one of the vibrations of the imperfect consonance; as appears by supposing $m-1$ pulses of AB or ab to be so intermitted as to leave only single equidistant pulses in Fig. 23, 24, 25, 26.

Pl. XII. Fig. 34. Now let any different multiples of AB and ab, as $3AB$ and $2ab$, or

(*d*) Defin. IV. Sect. VI.

AD and ac be the vibrations of the imperfect consonance; and if AB were$=ab$, then would $3AB$ or AD be to $2ab$ or $ac :: 3:2$, and all the short cycles of the vibrations, AD, ac would be perfect, or their exterior pulses G and g, N and n, &c, would be coincident, as in Fig. 33: because $2\times 3AB$ or $2AD$ or $AD+DG$ would then $= 3\times 2ab$ or $3ac$ or $ac+ce+eg$.

But AB in Fig. 34 being bigger than ab, the multiple $6AB$ or AG is also bigger than the equimultiple $6ab$ or ag; and so the Exterior pulses G and g, N and n, &c, of the short cycles, which pulses were coincident before, are now separated by some of the alternate lesser intervals, Gg, Nn, &c, of the pulses of AB and ab (e): the distances of the pulses G and g, N and n, &c, from A and a, being equimultiples of AB and ab.

For the like reason the Interior pulses, c, e, &c, of the imperfect short cycles are also separated from the pulses of AB and ab (denoted by round points when different from those of AD and ac) by some of their alternate lesser intervals.

Hence the dislocations of the pulses in all the imperfect short cycles, are some of the alternate lesser intervals of the pulses of Ab and ab.

For though we began the first short cycle from two coincident pulses A, a, yet the argument is the same if we suppose them separated by

(e) Pro. VII. coroll. 3.

by any one of the alternate leſſer intervals; or begin to count the vibrations of the conſonance from any two pulſes of AB and ab, as Q and r, whoſe diſtances from the next periodical point or coincident pulſes Z, η, are proportional to the vibrations AB, ab, that is, whoſe interval Qr is an alternate leſſer interval of their pulſes (f).

For ſince the ſeveral lengths QX and ry, $X\Delta$ and $y\varepsilon$, ΔK and $\varepsilon\lambda$, &c, of the ſubſequent ſhort cycles, are proportional to AB and ab, the remaining diſtances XZ and $y\eta$, ΔZ and $\varepsilon\eta$, KZ and $\lambda\eta$, &c, are alſo proportional to AB and ab; which ſhews that the diſlocations of the exterior pulſes X and y, Δ and ε, K and λ, &c, and of the interior too, are conſtantly ſome of the alternate leſſer intervals of the pulſes of AB and ab. And thus the period of the leaſt diſlocations of the pulſes of the imperfect conſonance, or of the leaſt imperfections in its ſhort cycles, is conſtantly the ſame as that of the pulſes of AB and ab.

Fig. 35 ſhews the ſame thing, when $2AB$ and $3ab$, or AC and ad, are different multiples of AB and ab, whoſe pulſes make long ſimple cycles; and when they make periods, the like is evident by inſpection of the pulſes about the periodical points X, Y, in Fig. 24, 25, 26, ſuppoſing the proper numbers of pulſes to be intermitted.

And univerſally, the leaſt terms of any perfect ratio being m and n, the periods of imperfect

(f) Prop. VII.

perfect unisons whose vibrations are AB and ab, will be changed into periods of the same length of an imperfect sharp consonance whose vibrations are mAB and nAb by intermitting $m-1$ pulses of AB and $n-1$ pulses of ab; or into equal periods of an imperfect flat consonance whose vibrations are mab and nAB, by intermitting $m-1$ pulses of ab and $n-1$ pulses of AB, so as to leave equidistant pulses at larger intervals for the pulses of the resulting consonance.

For tho' some of the alternate lesser intervals of the pulses of the imperfect unisons are destroyed by those intermissions, yet the remaining pulses continue in their own places and make periods of the same length as the whole number of pulses did before. Q. E. D.

Coroll. 1. With respect to the perfect consonance whose vibrations are mAB, and nAB, the former imperfect consonance of mAB and nab is tempered sharp by the tempering ratio nAB to nab (g), and the latter imperfect consonance of mab and nAB is tempered flat by the same ratio mAB to mab of the vibrations AB, ab of the imperfect unisons, whose interval is therefore the temperament of both those imperfect consonances.

And the same might be said with respect to this other perfect consonance of the vibrations mab and nab, whose interval is the same as that of the former perfect consonance, the perfect ratio being the same in both.

Coroll.

(g) Sect. II. Art 5 and 6.

Coroll. 2. The lengths of the perfect cycles of those perfect consonances are $mnAB$ and $mnab$; (because $mAB : nAB :: m : n :: mab : nab$;) and $mnAB$ being the greater of the two, is therefore the whole length of the imperfect short cycle of either of the foregoing tempered consonances.

Coroll. 3. Consequently the imperfect short cycle of any imperfect consonance contains equal numbers of the slower and quicker vibrations AB, ab of the imperfect unisons from whence it is derived.

Coroll. 4. The same multiples of the vibrations of imperfect unisons, will be the vibrations of other imperfect unisons, whose period is the same multiple of the period of the given unisons (*h*), and whose interval is the same too at a different pitch; because the ratio of the vibrations is the same (*i*).

LEMMA.

Pl. XII. Fig. 36. *The logarithms of small ratios,* a o *to* b o, c o *to* d o, *whose terms have a common half sum,* s o, *are very nearly proportional to the differences of the terms of each ratio.*

For by the supposition the point *s* bisects both *ab* and *cd*, the differences of those terms. And if any hyperbola described with the center *o* and

E 3 rectangular

(*h*) Prop. VII. coroll. 4. (*i*) Sect. I. Art. 10.

rectangular asymptotes *so*, *oq*, cuts the perpendiculars erected at *s*, *a*, *b*, *c*, *d*, in *t*, *e*, *f*, *g*, *h*, it is well known that the areas *abfe*, *cdhg*, arithmetically expressed, are hyperbolic logarithms of the ratios *ao* to *bo*, *co* to *do*; and that these logarithms are proportional to any other logarithms of the same ratios. And those areas *abfe*, *cdhg*, differ very little from the trapeziums *ablk*, *cdnm*, cut off by the tangent *ptq* at the point *t* in the middlemost perpendicular *ts*: because the common bases *ab*, *cd*, or differences of the terms *ao* and *bo*, *co* and *do*, are supposed to be very small in comparison to the terms themselves.

It appears then that the logarithm of the ratio *ao* to *bo* is to the logarithm of *co* to *do*, as the area *abfe* to the area *cdhg*, or very nearly as the trapezium *ablk* to the trapezium *cdnm*; or (because *s t* is the mean altitude of both) as the rectangle under *ab* and *st* to the rectangle under *cd* and *st*, or as *ab* to *cd*. Q. E. D.

Coroll. 1. The logarithms of small ratios, *ao* to *bo*, *ao* to *co*, which have a common term *ao*, are also very nearly proportional to the differences of their terms; but not so nearly as if the terms had a common half sum.

For the logarithms of the ratios *ao* to *bo*, *ao* to *co* are proportional to the areas *abfe*, *acge*, or very nearly to the trapeziums *ablk*, *acmk*, or to the rectangles under their bases *ab*, *ac* and their mean altitudes, or nearly to the bases themselves: because the ratio of the mean altitudes is very small in comparison to that of the bases.

Coroll.

Coroll. 2. Pl. XII. Fig. 37. The logarithms of any ſmall ratios $ao : bo$, $co : do$ are very nearly in the ratio of $\frac{ab}{ao}$ to $\frac{cd}{co}$, or of $\frac{ab}{bo}$ to $\frac{cd}{do}$, compounded of the direct ratio of the differences of the terms of the propoſed ratios, and the inverſe ratio of their homologous terms.

For ſuppoſing $ao : eo :: co : do$, theſe ratios have the ſame logarithm. Whence the logarithms of the propoſed ratios, $ao : bo$, $co : do$, or $ao : eo$, are as ab to ae by cor. 1, or as $\frac{ab}{ao}$ to $\frac{ae}{ao}$ or $\frac{cd}{co}$, becauſe $ae : ao :: cd : co$ by the ſuppoſition. The ſecond part may be proved in like manner by taking $fo : ao :: co : do$.

Coroll. 3. If a and b be the terms of any ſmall ratio whoſe logarithm is c and $\frac{q}{p}c$ be any part or parts of it; taking $s = \frac{a+b}{2}$ and $d = \frac{a-b}{2}$, $s + \frac{q}{p}d$ to $s - \frac{q}{p}d$ is the ratio whoſe logarithm is $\frac{q}{p}c$ very nearly.

For the terms of both thoſe ratios have a common half ſum s, and ſince $s+d=a$ and $s-d=b$, the difference of the terms a and b is $2d$, and that of the terms $s + \frac{q}{p}d$ and $s - \frac{q}{p}d$ is $\frac{2q}{p}d$. Whence by the lemma, the logarithm of a to b is to the logarithm of $s + \frac{q}{p}d$ to $s - \frac{q}{p}a$ $:: 2d : \frac{2q}{p}d :: 1 : \frac{q}{p} :: c : \frac{q}{p}c$, and c being the logarithm of a to b, $\frac{q}{p}c$ is the logarithm of $s+\frac{q}{p}d$ to $s-\frac{q}{p}d$.

Coroll. 4. Hence as musical intervals are proportional to the logarithms of the ratios of the single vibrations of the terminating sounds (k), if any part or parts of a comma c denoted by $\frac{q}{p}c$, be the interval of imperfect unisons, the ratio of the times of their single vibrations will be $161p+q$ to $161p-q$.

For the comma c being the interval of two sounds whose single vibrations are as 81 to 80 (l), by substituting 81 for a and 80 for b in the last corollary, we have $s=\frac{161}{2}$, $d=\frac{1}{2}$ and $s+\frac{q}{p}d$ to $s-\frac{q}{p}d :: \frac{161}{2}+\frac{q}{p}\times\frac{1}{2} : \frac{16}{2}-\frac{q}{p}\times\frac{1}{2} :: 161p+q : 161p-q$, the ratio of the single vibrations belonging to the interval $\frac{q}{p}c$, very nearly (m).

This

(k) Sect. I. Art. 11. (l) Sect. II. Art. 4.

(m) And conversely, if the ratio of the times of the single vibrations of imperfect unisons be V to v, their interval is $\frac{V-v}{V+v}\times 161c$. For supposing $V : v :: 161p+q : 161p-q$, p and q being indeterminate numbers; or $V=161p+q$ and $v=161p-q$; then $V-v=2q$ and $V+v=161\times 2p$, and $\frac{V-v}{V+v}=\frac{q}{161p}$. Whence $\frac{V-v}{V+v}\times 161c=\frac{q}{p}c$, the interval belonging to the ratio $161p+q : 161p-q$, or $V:v$, by coroll. 4.

This ratio approaches ſurpriſingly near to the truth, as will appear by an example. Let $\frac{q}{p}c = \frac{1}{4}c$, or $p=4$ and $q=1$, then $161p+q$: $161p-q$: : as 645 : 643. Now by the Tables of Logarithms,

The log. $\frac{81}{80}$	$= 0.00539.50319$
$\frac{1}{4}$ of it	$= 0.00134.87580$
the log. $\frac{645}{643}$	$= 0.00134.87417$
the difference	$= 0.00000.00163$

and the logarithm 539.50319 divided by the difference 163 gives the quotient 330984, which ſhews that $\frac{1}{4}c$ deduced from the ratio 645:643 differs from the truth but by $\frac{1}{330984}$ part of a comma; a degree of exactneſs abundantly ſufficient for every purpoſe in harmonics.

Coroll. 5. The times of the ſingle vibrations of imperfect uniſons being V and v and their interval $\frac{q}{p}c$; $V : v :: 161p+q : 161p-q$ and the period of their pulſes is $\frac{161p-q}{2q}V$ or $\frac{161p+q}{2q}v$ (*n*).

Likewiſe the Vibrations of other imperfect uniſons being V' and v' and their interval $\frac{q'}{p}c$; $V' : v' :: 161p+q' : 161p-q'$ and the period of their pulſes is $\frac{161p-q'}{2q'}V'$ or $\frac{161p+q'}{2q'}v'$.

Coroll.

(*n*) See Sect. VI. Defin. III.

Coroll. 6. Hence if the intervals of two consonances of imperfect unisons be equal, or $q=q'$, the periods of their pulses have the ratio of their slower or quicker vibrations, V to V', or v to v', which ratios are therefore the same (*o*).

Coroll. 7. The Ultimate Ratio of the periods of imperfect unisons is compounded of the ratio of their slower or quicker vibrations directly and of that of their intervals inversely, and so it is $\frac{V}{q}$ to $\frac{V'}{q'}$, or $\frac{v}{q}$ to $\frac{v'}{q'}$.

For supposing the quantities p, v, v' in coroll. 5. to be variable and p to increase to infinity in any finite time, the intervals $\frac{q}{p}c$, $\frac{q'}{p}c$, will decrease and vanish in the ratio of q to q' first given; the ratio $161p-q$ to $161p-q'$ will also decrease and vanish in the ratio of equality; and therefore the Ultimate Ratio with which the increasing periods $\frac{161p-q}{2q}V$ and $\frac{161p-q'}{2q'}V'$ became infinite at the end of the given time, and vanished into innumerable short cycles of perfect unisons, is $\frac{V}{q}$ to $\frac{V'}{q'}$, or, by a like argument, $\frac{v}{q}$ to $\frac{v'}{q'}$.

Coroll. 8. Hence if two consonances of imperfect unisons have a common sound or vibration $V=V'$, or $v=v'$, the ultimate ratio of their periods is q' to q, the inverse ratio of their intervals;

(*o*) This agrees with Cor. 4. Prop. VII.

intervals; and consequently is the inverse ratio of the differences of their vibrations (p).

Coroll. 9. If the two slower or the two quicker vibrations of two consonances of imperfect unisons have the ratio of their intervals, the periods of their pulses are ultimately equal. For if $V : V' :: q : q'$, then $\frac{V}{q} : \frac{V'}{q'} :: 1 : 1$, which is the ultimately ratio of the periods: and the like argument is applicable to the ratio $v : v'$.

PROPOSITION IX.

If the interval of two sounds whose perfect ratio is m *to* n, *be increased or diminished by* $\frac{q}{p}c$, *and the times of the complete vibrations of the base and treble of either of these consonances be* Z *and* z *and the period of its least imperfections be* P, *then in*

$$Cas.\ 1,\ P = \frac{161p - q}{2q} \times \frac{Z}{m} \ or\ \frac{161p + q}{2q} \times \frac{z}{n},$$

$$Cas.\ 2,\ P = \frac{161p + q}{2q} \times \frac{Z}{m} \ or\ \frac{161p - q}{2q} \times \frac{z}{n}.$$

Pl. XII. Fig. 38. For if AV and av, or V and v be the times of the complete vibrations of imperfect unisons whose interval is the temperament $\frac{q}{p}c$, then $V : v :: 161p + q : 161p - q$ (q) and the period of their pulses, or of their least imper-

(p) Coroll. 1. and Sect. 1. Art. 11,
(q) Coroll. 4. Lemma.

imperfections, is $\frac{161p-q}{2q}V = \frac{161p+q}{2q}v$ (r) and has the ſame length as the period of the leaſt imperfections in a ſharp conſonance whoſe vibrations are mV and nv, or in a flat conſonance whoſe vibrations are mv and nV; both conſonances being derived from the perfect one whoſe vibrations are mV and nV, or mv and nv (s).

Hence in Caſ. 1. taking $Z=m$V and $z=nv$, we have $\frac{Z}{m}=V$ and $\frac{z}{n}=v$; which values being ſubſtituted for V and v in the period of imperfect uniſons, give $P=\frac{161p-q}{2q}\times\frac{Z}{m}=\frac{161p+q}{2q}\times\frac{z}{n}$. And in Caſ. 2. (Z and z being ſuppoſed indeterminate) taking $Z=mv$ and $z=n$V, we have $\frac{Z}{m}=v$ and $\frac{z}{n}=V$; which being ſubſtituted for v and V in the ſaid period of imperfect uniſons, give $P=\frac{161p+q}{2q}\times\frac{Z}{m}=\frac{161p-q}{2q}\times\frac{z}{n}$. Q. E. D.

The value of P in Caſe 2 is deducible from its value in Caſe 1, only by changing the ſign of q; that is, by ſuppoſing $\frac{q}{p}c$ to be the negative or flat temperament, as it really is when $+\frac{q}{p}c$ is the ſharp one. And thus one expreſſion of the value of P might have ſerved both caſes of the propoſition, but two are more diſtinct for future uſe.

Coroll.

(r) Sect. VI. Defin. III. or Cor. 5. Lemma.
(s) Prop. VIII. Corol. 1.

Coroll. 1. Hence if any two imperfect consonances have Z and Z' for the times of the complete vibrations of their bafes, z and z' for thofe of their trebles, $\frac{q}{p}c$ and $\frac{q'}{p}c$ for their temperaments, whether fharp or flat, or one of each fort, m and m' for the major, and n and n' for the minor terms of the perfect ratios; the Ultimate ratio of their periods is $\frac{Z}{qm}$ to $\frac{Z'}{q'm'}$, or $\frac{z}{qn}$ to $\frac{z'}{q'n'}$: The proof of which is the fame as was given for the Ultimate ratio of the periods of imperfect unifons in Coroll. 7. to the Lemma.

Coroll. 2. Hence when the temperaments are equal and the major terms the fame, the periods of the leaft imperfections have ultimately the ratio of the fingle vibrations of the bafes.

Coroll. 3. When the bafes are the fame, the periods have ultimately the inverfe ratio of the temperaments and major terms jointly.

Coroll. 4. When the bafes and major terms are the fame, the periods have ultimately the inverfe ratio of the temperaments.

Coroll. 5. When the bafes and temperaments are the fame, the periods have ultimately the inverfe ratio of the major terms.

Coroll. 6. All thofe corollaries are applicable to the trebles and the minor terms, only by reading trebles inftead of bafes and minor terms inftead of major; and then, as before they had no dependence on the trebles and minor terms,

fo

ſo now they have none upon the baſes and major terms.

PHÆNOMENA OF BEATS.

If a conſonance of two ſounds be uniform, without any beats or undulations, the times of the ſingle vibrations, of its ſounds have a perfect ratio; but if it beats or undulates, the ratio of the vibrations differs a little from a perfect ratio, more or leſs according as the beats are quicker or ſlower.

Change the firſt and ſmalleſt ſtring of a violoncello for another about as thick as the ſecond, that their ſounds having nearly the ſame ſtrength may beat ſtronger and plainer. Then ſkrew up the firſt ſtring; and while it approaches gradually to an uniſon with the ſecond, the two ſounds will be heard to beat very quick at firſt, then gradually ſlower and ſlower, till at laſt they make an uniform conſonance without any beats or undulations. At this juncture either of the ſtrings ſtruck alone, by the bow or finger, will excite large and regular vibrations in the other, plainly viſible to the eye; which ſhews that the times of their ſingle vibrations are equal (*t*).

Alter the tenſion of either ſtring a very little, and their ſounds will beat again. But now the motion

(*t*) Sect. 1. Art. 1.

motion of one ſtring ſtruck alone makes the other only ſtart, but excites no regular vibrations; a plain proof that they are not iſochronous. And while the ſounds of both are drawing out with an even bow, not only an audible but a viſible beating and irregularity is obſervable in the vibrations, which in the former caſe were free and uniform.

Meaſure the length of either ſtring between the nut and bridge, and, when they are perfect uniſons, at the diſtance of $\frac{1}{3}$ of that length from the nut mark that ſtring with a ſpeck of ink. Then placing the edge of your nail on the ſpeck, or very near it, and preſſing it to the finger-board, upon ſounding the remaining $\frac{2}{3}$ with the other ſtring open, you will hear an uniform conſonance of V^ths^, whoſe ſingle vibrations have the perfect ratio of 3 to 2 (*u.*) But upon moving your nail a little downwards or upwards, that ratio will be a little increaſed or diminiſhed; and in both caſes the imperfect V^ths^ will beat quicker or ſlower according as that perfect ratio is more or leſs altered.

The Phænomena are the ſame when the parts of the ſtring have any other perfect ratio; except that the beats of the ſimpler concords are plainer than thoſe of the leſs ſimple and theſe plainer than thoſe of the diſcords, which being very quick are not eaſily diſtinguiſhed from the uniform roughneſs of perfect diſcords.

The

(*u*) Sect. 1. Art. 7.

The ſounds of an organ being generally more uniform than any other, their beats are accordingly more diſtinct, and are perfectly iſochronous when the blaſt of the bellows is ſo uniform as not to alter the vibrations of either ſound.

Beats and undulations when every thing elſe is ſilent, are alſo pretty plain upon the harpſichord, eſpecially while the ſounds are vaniſhing.

Quicker undulations are beats, and are remarkably diſagreeable in a concert of ſtrong, treble voices, when ſome of them are out of tune; or in a ring of bells ill tuned, the hearer being near the ſteeple; or in a full organ badly tuned: nor can the beſt tuning wholly prevent that diſagreeable battering of the ears with a conſtant rattling noiſe of beats, quite different from all muſical ſounds, and deſtructive of them, and chiefly cauſed by the compound ſtops called the Cornet and Seſquialter, and by all other loud ſtops of a high pitch, when mixed with the reſt. But if we be content with compoſitions of uniſons and octaves to the Diapaſon, whatever be the quality of their ſounds, the beſt manner of tuning will render the noiſe of their beats inoffenſive if not imperceptible. Theſe are the general phœnomena of beats, whoſe theory I am going to explain.

PRO-

PROPOSITION X.

An imperfect consonance makes a beat in the middle of every period of its least imperfections, and so the time between its successive beats is equal to the periodical time of its least imperfections.

Pl. XI. Fig. 23 to 27. 34, 35. Any simple cycle or any period of the pulses of imperfect unisons, contains one more of the quicker than of the slower vibrations (*x*), and the short cycle of any imperfect consonance contains equal numbers of the quicker and slower vibrations of the imperfect unisons (*y*). Consequently after taking away the greatest equal numbers of short cycles, that can be taken from both ends of the simple cycle or the period of the imperfect unisons, some part of another short cycle or two, as consisting of unequal numbers of the quicker and slower vibrations of the imperfect unisons, will always remain in the middle of the cycle or period. And this part, by interrupting the succession of the short cycles, wherein the harmony of the consonance consists, interrupts its harmony and causes the noise which is called a beat: especially as the interruption is made where the short cycles on each

(*x*) Prop. VII. coroll. 1.
(*y*) Prop. VIII. coroll. 3.

 side

side of it are the most imperfect and inharmonious. Therefore the time between the successive beats, made in the middle of each period or simple cycle of the pulses of the imperfect unisons, or of the least imperfections of the consonance (z), is equal to the time of this period.

And the cause of the beats of imperfect unisons is a like interruption of the succession of their short cycles, in the middle of every period or simple cycle of their pulses, where they are most imperfect and inharmonious. Q. E. D.

Coroll. The time between the successive beats of an imperfect consonance is the same as the periodical time of its Greatest Imperfections.

PROPOSITION XI.

If the interval of two sounds whose perfect ratio is m *to* n, *be increased or diminished by the temperament* $\frac{q}{p}$c (a), *and* β *be the number of beats made by either of those consonances while its base is making* N, *and its treble* M *complete vibrations; then in*

$$Cas.\ 1,\ \beta = \frac{2q}{161p - q}\, mN, \text{ or } \frac{2q}{161p + q}\, nM,$$

$$Cas.\ 2,\ \beta = \frac{2q}{161p + q}\, mN, \text{ or } \frac{2q}{161p - q}\, nM.$$

For if the time between the successive beats of either consonance be called P, and the time of

(z) Prop. VIII. (a) See Lemma cor. 4. p. 72.

of a complete vibration of its base be Z and that of its treble z; the time of their beating and vibrating will be constantly measured by $\beta P = NZ$ or Mz. Hence $\beta = \frac{NZ}{P}$ or $\frac{MZ}{P}$ and since the time P is equal to the period of the least imperfections of the consonance (*b*), by substituting its values in Prop. IX, we have in Cas. 1. $\beta = NZ \times \frac{2qm}{161p-q.Z} = \frac{2qmN}{161p-q}$, and so of the other values of β. Q. E. D.

Coroll. 1. Hence if any two imperfect consonances have Z and Z′ for the times of the single vibrations of their bases, z and z' for those of their trebles, $\frac{q}{p}c$ and $\frac{q'}{p}c$ for their temperaments, whether flat or sharp, or one of each sort, m and m' for the major, n and n' for the minor terms of the perfect ratios, N and N′ for the numbers of complete vibrations made by the bases, and M and M′ for those made by the trebles in any given time; the ultimate ratio of the numbers of their beats, made in that time, will be $qmN : q'm'N'$, or $qnM : q'n'M'$, or $\frac{qm}{Z} : \frac{q'm'}{Z'}$, or $\frac{qn}{z} : \frac{q'n'}{z'}$.

The manner of proving the two first ratios has been shewn before (*c*), and the given time being constantly $NZ = N'Z' = Mz = M'z'$, we have $N : N' :: \frac{1}{Z} : \frac{1}{Z'}$, which ratios compounded with $qm : q'm'$ give $qmN : q'm'N' :: \frac{qm}{Z} : \frac{q'm'}{Z'}$.

F 2 Like-

(*b*) Prop. x. (*c*) Lemma cor. 7.

Likewiſe $M : M' :: \frac{1}{z} : \frac{1}{z'}$ which ratios compounded with $qn : q'n'$ give $qnM : q'n'M' :: \frac{qn}{z} : \frac{q'n'}{z'}$.

Coroll. 2. Hence, when the temperaments are equal and the major terms the ſame, the beats of the conſonances, made in a given time, have ultimately the inverſe ratio of the ſingle vibrations of the baſes.

Coroll. 3. When the baſes are the ſame, the beats have ultimately the ratio of the temperaments and major terms jointly: And therefore when the baſes and beats are the ſame, the temperaments have ultimately the inverſe ratio of the major terms.

Coroll. 4. When the baſes and major terms are the ſame, the beats have ultimately the ratio of the temperaments.

Coroll. 5. When the baſes and temperaments are the ſame, the beats have ultimately the ratio of the major terms.

Coroll. 6. All theſe corollaries are applicable to the trebles and minor terms, by reading trebles inſtead of baſes and minor terms inſtead of major: and then they have no dependence on the baſes and major terms, as in the former caſes they had none upon the trebles and minor terms: which abſent terms may therefore in both caſes have any magnitudes whatever without altering the ratio of the beats.

Coroll.

Coroll. 7. Things remaining as in the propofition, we have in

Caf. 1. $Z : z :: m + \frac{\beta}{N} : n :: m : n - \frac{\beta}{N}$.

Caf. 2. $Z : z :: m - \frac{\beta}{N} : n :: m : n + \frac{\beta}{N}$.

For by the Prop. in Caf. 1. $\beta = \frac{2qmN}{161p-q}$, whence $\frac{\beta}{N} : m :: 2q : 161\,p-q$ and *compofitè* $m + \frac{\beta}{N} : m :: 161p+q : 161p-q$, either of which ratios being the tempering ratio and m to n the perfect one, the imperfect ratio is plainly $m + \frac{\beta}{N} : n :: Z : z$. And a like refolution of the other values of β in the propofition gives the other proportions.

Scholium 1.

To fhew that the ultimate ratio of the beats or the periods of imperfect confonances, when ufed inftead of the exact ratio, can produce no fenfible difference in the Harmony.

1. The temperaments of any two confonances being $\frac{q}{p}c$ and $\frac{q'}{p}c$, the difference between the exact and the ultimate ratio of their beats, made in any given time, is the ratio $161\,p \mp q$ to $161\,p \mp q'$; where the fign of q or q' is negative if the refpective temperament be fharp, or affirmative if flat.

For that ratio compounded with the exact ratio of the beats, which is $\frac{q\,m\,N}{161\,p\mp q}$ to $\frac{q'\,m'\,N'}{161\,p\mp q'}$ (*d*), makes their ultimate ratio $q\,m\,N$ to $q'\,m'\,N'$ (*e*).

2. Now the magnitude of the ratio $161\,p\mp q$ to $161\,p\mp q'$, like that of all ratios, being greatest or least according as the difference of its terms is greatest or least in proportion to the terms themselves; it will follow, that in the most harmonious system of sounds hereafter determined (*f*), the ultimate ratio of the beats of any two concords cannot differ from their exact ratio by any ratio greater than $362\frac{5}{8}$ to $361\frac{5}{8}$, or less than 2901 to 2900.

For the temperaments $\frac{q}{p}\,c$, $\frac{q'}{p}\,c$ of any two concords in that system, have no other values than a couple of these, $\frac{5}{18}c$, $\frac{3}{18}c$, and $\frac{2}{18}c$ (*f*). Where p being 18, the greatest magnitude of the said ratio $161\,p\mp q$ to $161\,p\mp q'$ is $161\times18+3$ to $161\times18-5$, or $362\frac{5}{8}$ to $361\frac{5}{8}$, and the least magnitude of it is $161\times18+3$ to $161\times18+2$, or 2901 to 2900.

3. Hence the number of beats in either term of any ultimate ratio in that system, cannot differ from the number of them in the corresponding term of the exact ratio, by above $\frac{1}{361}$ part of that first number: and therefore not by

(*d*) Prop. XI.
(*e*) Prop. XI. cor. 1.
(*f*) Prop. XVI. Schol. 2. Art. 10 and 13.

by a ſingle beat when that number is leſs than 361.

For let a to e be an ultimate ratio which exceeds the correſponding exact ratio by the greateſt difference $362\frac{5}{8}$ to $361\frac{5}{8}$. Then by ſubſtracting this difference, and neglecting the fractions, the exact ratio is $361a$ to $362e$, that is, a to $e+\frac{1}{361}e$, or $a-\frac{1}{362}a$ to e.

4. Now let two v^ths, or any two concords of the ſame name, near the middle of the ſcale of a good organ, have the ſame baſe and different trebles; and ſuppoſe them ſo nicely tempered, that in a given time one of the v^ths ſhall make 362 beats and the other 361. This indeed is extremely difficult to execute, the numbers of beats being ſo large. But ſuppoſing it done, my opinion is (from my own experience in ſmaller numbers) that the moſt critical ear could not diſtinguiſh the leaſt difference in the harmony of thoſe v^ths, or in the rate of their beating: no not if the ratio of the beats were much greater than 362 to 361: And if it could not, without doubt the theory of ultimate ratios is ſufficiently accurate for determining and adjuſting the Harmony of the beſt ſyſtem of ſounds. Becauſe it will be ſhewn hereafter, that the beſt method of tuning any ſyſtem, is to adjuſt every v^th to the number of beats it ſhould make in that ſyſtem.

5. In leſs harmonious ſyſtems, the difference between the exact and the ultimate ratio is ſome-

thing greater than 362 to 361; as $322\frac{1}{2}$ to $321\frac{1}{2}$ in the syftem of mean tones (*g*); but ftill not fo great in any tolerable fyftem as to affect the moft critical ear: and what has been proved of beats holds true of Periods, the ratio of the periods being the inverfe ratio of the numbers of beats made in any given time.

6. Therefore the ultimate ratios of beats and periods ought to be ufed in harmonics, their terms being always fimpler than thofe of the exact ratios, as appears by comparing Prop. IX and XI with their corollaries.

For inftance in the fyftem of equal harmony, the temperament of the v^th^ is $\frac{-5}{18}c$, and of the vi^th^ is $\frac{+3}{18}c$, whence if their bafes be the fame the exact ratio of their beats, made in any given time, is $361\frac{7}{8}$ to $362\frac{7}{8}$ by Prop. XI; but their ultimate ratio is that of equality by coroll. 3, which is fimpler, and the harmony of the concords not fenfibly different (*h*).

Scholium 2.

To fhew that the theory of beats agrees with experiments.

1. Pl. I. Fig. 3. The exponent of the time of a fingle vibration of any given found, as *c*, in

(*g*) Prop. 2. coroll.
(*h*) Art. 4. of thefe.

in a given ſyſtem of perfect conſonances may be changed into 1 by dividing every exponent by that of the given ſound, which changes them to thoſe in Pl. XII. Tab. 1. without altering their proportions.

Then if the ſound whoſe exponent is 1, be a little altered to γ either higher or lower, the numbers of beats made in any given time by the ſeveral imperfect conſonances of γ with every one of the other ſounds, will be proportional to the Denominators of their exponents.

For when γ is flatter than c, all the intervals above c are increaſed by the common temperament $c\gamma = \frac{q}{p} c$ in Prop. XI, where in Caſe 1 the number of beats made by any given conſonance γd, while its baſe γ is making N vibrations, is $\frac{2q}{161p-q} m\text{N}$. And all the perfect intervals below c being diminiſhed by $c\gamma$, in Caſe 2 the number of beats of any given conſonance γB, while its treble γ is making M vibrations, is $\frac{2q}{161p-q} n\text{M}$.

Here the numbers N, M of the vibrations of γ made in any given time are equal, and $\frac{2q}{161p-q}$ being the ſame in both caſes, the beats of γd are to thoſe of γB as m to n, that is as the major term 9 of the perfect ratio 9 to 8 belonging to to $\gamma\, d$, is to the minor term 15 of the perfect ratio 15 to 16 belonging to γ B; and those terms are the Denominators of the exponents of *d* and B, the treble of the former and the base of the latter consonance.

And since the beats of $\gamma\, d$ and γ B are as 9 to 15, and by the same demonstration those of γ B and $\gamma\, e$ as 15 to 5, *ex æquo* the beats of $\gamma\, d$ and $\gamma\, e$, having the same base, are as 9 to 5; which terms are the Denominators of the exponents of the trebles *d*, *e*.

And by the like proportions the beats of γ B, γ A, which have the same treble, are as 15 to 5, the Denominators of the exponents of the bases B, A.

And when γ is sharper than *c*, the two theorems above are changed to these $\frac{2q}{161p+q}m$ N and $\frac{2q}{161p+q}n$ M, and the demonstration goes on as before. Q. E. D.

2. In Tab. 2 and 3 each series of fractions, being a geometrical progression in the ratio 2 to 1, are the exponents of the single vibrations of successive VIII$^{\text{ths}}$, and are severally deduced from the exponents of the bases of as many given concords AC, AD, AF, AC※, AE, AF※.

Hence in Tab. 2. the beats which the treble of any imperfect Minor consonance, AC, AD or AF, makes in a given time with its base and with every 8$^{\text{th}}$ below it and as many 8$^{\text{ths}}$ above it it as reſult from a continual biſection of the Numerator of its exponent, are all iſochronous. But the beats which that treble makes with the ſucceſſive 8ths ſtill higher are continually doubled in any given time.

TAB. 2.

$\frac{12}{5}$	$\frac{6}{5}$	1	$\frac{3}{5}$	$\frac{3}{10}$	$\frac{3}{20}$
A′	A	C	*a*	*a′*	*a″*
$\frac{8}{3}$	$\frac{4}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{6}$
A′	A	D	*a*	*a′*	*a″*
$\frac{16}{5}$	$\frac{8}{5}$	1	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{5}$
A′	A	F	*a*	*a′*	*a″*

For the major term of the perfect ratio of any Minor conſonance is an Even number (*i*) and is the Numerator of the exponent of its baſe (that of its treble being reduced to 1,) and when that numerator is reduced to an odd number by continual biſections, this odd number is the conſtant numerator of the exponents of all the ſuperior 8ths, whoſe denominators muſt therefore be continually doubled, which doubles the beats by Art. 1. But the doubling the numerators of the exponents of the inferior 8ths alters not their given denominator, as being an odd number, nor conſequently the beats.

Tab. 3.

(*i*) Sect. 2. Art. 1. Table.

Tab. 3. The beats which the treble of any imperfect Major consonance, AC※, AE or AF※, makes with its base, in any given time, and with every 8th above it and as many 8ths below it as result from a continual bisection of the Denominator of its exponent, if an Even number, are continual proportionals in the ratio of 2 to 1; and the beats of that treble with every 8th still lower are isochronous. But if the Denominator of the given base be an odd number, the beats which its treble makes with it and every 8th below it are all isochronous.

TAB. 3.

$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{4}$	1	$\frac{5}{8}$	$\frac{5}{16}$
A″	A′	A	C※	*a*	*a′*
$\frac{6}{1}$	$\frac{3}{1}$	$\frac{3}{2}$	1	$\frac{3}{4}$	$\frac{3}{8}$
A″	A′	A	E	*a*	*a′*
$\frac{20}{3}$	$\frac{10}{3}$	$\frac{5}{3}$	1	$\frac{5}{6}$	$\frac{5}{12}$
A″	A′	A	F※	*a*	*a′*

For the major term of the perfect ratio of any Major consonance is an Odd number (*k*), and is the Numerator of the exponent of its base (that of its treble being 1) and its denominator continually doubled gives the successive denominators

(*k*) Sect. 2. Art. 1. Table.

minators of the exponents of all the aſcending 8ths, and continually halved, if it be an even number, gives thoſe of the deſcending 8ths, till it be reduced to an Odd Number; which continues to be the denominator of all the exponents ſtill lower. But if the Denominator of the given baſe be Odd, it is itſelf the denominator of the exponents of all the inferior 8ths. Therefore the law of the beats is evident by Art. 1.

4. Tab. 1. 2. 3. Hence any two imperfect conſonances which compoſe a perfect 8th, will beat equally quick, if the minor conſonance be below the major; but if the minor be above the major, it will beat twice as quick as the major: the denominators of the exponents of the baſe and treble of the 8th being equal in the firſt caſe and as 2 to 1 in the ſecond.

5. Theſe examples are ſufficient to ſhew the agreement of the theory of beats with the eaſieſt experiments, as requiring no more to be done in many inſtances than to examine by the ear whether the ſucceſſive 8ths, as A′, A, *a*, *a*′, &c, throughout the ſcale of the organ or harpſichord be quite perfect, and if not, to make them ſo. For the conſonances which compoſe thoſe 8ths being made imperfect, as they uſually are and ought to be, the ear will judge very well whether the beats of ſuch concords as by theory ought to be iſochronous, are really ſo or not when ſounded immediately after one another.

6. Theſe experiments attentively tried will be perceptible in ſome degree upon a ſingle ſtop of a good harpſichord, and very plainly upon the

the open diapaſon of a good organ; where the beats of the ſimpler concords about the middle of the ſcale will be very diſtinct and ſlow enough to be eaſily counted. The equal times of beating may be meaſured by a watch that ſhews ſeconds or a ſimple pendulum of any given length: and if the blaſt of the bellows be ſufficiently uniform, it may be queſtioned whether an 8th may not be tuned perfect or nearer to perfection by the iſochronous beats of a minor and major concord which compoſe it (*l*) than by the judgment of the moſt critical ear.

7. Pl. XII. Tab. I. Of conſonances which have a common ſound and a common temperament, the ſimpler generally beat ſlower than the leſs ſimple do; the denominators of the exponents of the ſimpler being generally ſmaller (*m*).

Scholium 3.

1. *Merſennus* and Mr. *Sauveur* are the only writers I know of that take any notice of the phyſical cauſe of the beats of conſonances. *Sauveur* imagined that they beat at every coincidence of their pulſes (*n*), and obſerving that he

(*l*) Art. 4. of theſe. (*m*) Sect. III. Art. 5.

(*n*) M. *Sauveur* ayant cherché la cauſe de ce Phenomene, a imaginé avec une extrême vraiſemblance, que le ſon des deux tuyaux enſemble devoit avoir plus de force, quand he could diſtinguiſh the beats pretty well when they went no quicker than 6 in one ſecond, and ſtill plainer when they went ſlower, he concluded that he could not perceive them at all when they went quicker than at that rate (*o*); and thence he inferred that octaves and other ſimple concords, whoſe vibrations coincide very often, are agreeable and pleaſant becauſe their beats are too quick to be diſtinguiſhed, be the pitch of the ſounds ever ſo low; and on the contrary, that the more complex conſonances whoſe vibrations coincide ſeldomer, are diſagreeable becauſe we can diſtinguiſh their ſlow beats; which diſpleaſe the ear, ſays he, by reaſon of the inequality of the ſound (*p*). And in purſuing this thought he found, that thoſe conſonances which beat faſter than 6 times in a ſecond, are the very ſame that muſicians treat as concords; and that others which beat ſlower are the diſcords; and he adds, that when a conſonance is a diſcord at a low pitch and a concord

quand leurs vibrations, après avoir été quelque temps ſeparées, venoient à ſe réunir et s'accordoient à frapper l'oreille d'un même coup. Hiſtoire de l'Acad. Royale des Sciences, année 1700, pag. 171. 8vo.

(*o*) Donc dans tous les accords où les vibrations ſe rencontreront plus de 6 fois par ſeconde, on ne ſentira point de battemens, et on les ſentira au contraire avec d'autant plus de facilité que les vibrations ſe rencontreront moins de 6 fois par ſeconde. ibid. pag. 176.

(*p*) Les battemens ne plaiſent pas à l'oreille, à cauſe de l'inégalité du ſon, et l'on peut croire avec beaucoup d'apparence, que ce qui rend les octaves ſi agréables, c'eſt qu'on n'y entend jamais de battemens. ibid. pag. 177.

concord at a high one, it beats ſenſibly at the former pitch but not at the latter (*q*).

2. As Mr. *Sauveur* appeals to numbers, let us ſee what evidence they produce. The tones and ſevenths major and minor being diſcords, muſt beat ſlower than 6 times in one ſecond by his own hypotheſis. Then let them beat but 4 or 5 times, and it will follow that the major IVth and minor 5th cannot beat above once in a ſecond.

For the lengths of the cycles of perfect conſonances to a common baſe, are proportional to the leſſer terms of the ratios of their vibrations (*r*), which being but 8 and 9 in the former diſcords and 32 and 45 in the latter (*s*), ſhew, that the latter muſt beat 4 or 5 times ſlower than the former, that is, as ſlow at leaſt as a clock that beats ſeconds.

But in ſounding the latter diſcords upon an Organ, Harpſichord or Violoncello, even at a low pitch, I find their beats are ſo quick that I cannot count them; or rather they do not beat at all, like imperfect conſonances, but only flutter,

(*q*) En ſuivant cette idée, on trouve que les accords dont on ne peut entendre les battemens, ſont juſtement ceux que les muſiciens traitent de conſonances, et que ceux dont les battemens ſe font ſentir, ſont les diſſonances; et que quand un accord eſt diſſonance dans une certaine octave, & conſonance dans une autre, c'eſt qu'il bat dans l'une, et qu'il ne bat pas dans l'autre. ibid. pag. 177.

(*r*) Sect. III. Art, 13.

(*s*) Table of perfect ratios, Sect. 2. Art. 1.

flutter, at a flower or quicker rate according to the pitch of the founds.

The truth is, this gentleman confounds the diftinction between perfect and imperfect confonances, by comparing imperfect confonances (*t*) which beat becaufe the fucceffion of their fhort cycles is periodically confufed and interrupted (*u*), with perfect ones which cannot beat, becaufe the fucceffion of their fhort cycles is never confufed nor interrupted.

3. The fluttering roughnefs abovementioned is perceivable in all other perfect confonances, in a fmaller degree in proportion as their cycles are fhorter and fimpler, and their pitch is higher, and is of a different kind from the fmoother beats and undulations of tempered confonances; becaufe we can alter the rate of the latter by altering the temperament, but not of the former, the confonance being perfect at a given pitch: And becaufe a judicious ear can often hear, at the fame time, both the flutterings and the beats of a tempered confonance, fufficiently diftinct from each other.

Scholium 4.

1. In all tempered fyftems the times of the fingle vibrations of moft of the confonances are incommenfurable quantities.

(*t*) Memoires de l'Acad. 1701, Syftême, general, Sect. XII, maniere de trouver le fon fixe. pag. 473. 8vo.

(*u*) Dem. of Prop. X.

In the syſtem of mean tones, for inſtance, the ſingle vibrations of the ſounds which terminate the tone are in the ratio of $\sqrt{5}$ to 2, the ſubduplicate of 5 to 4, as the mean tone is half the III^d^. Likewiſe the ſingle vibrations of V^ths^ tempered by a quarter of a comma, are in the ratio of $\sqrt[4]{5}$ to 1, the ſubquadruplicate of 5 to 1, as the interval of the V^ths^ is a quarter of the XVII^th^ or 2 VIII + III. For as $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{81}{80} = \frac{1}{5}$, ſo $V + V + V + V - c = XVII$; whence $V - \frac{1}{4}c + V - \frac{1}{4}c + V - \frac{1}{4}c + V - \frac{1}{4}c = XVII$. Laſtly the ratio of the vibrations of two ſounds whoſe interval is a quarter of a comma, is $\sqrt[4]{81}$ to $\sqrt[4]{80}$, or $\sqrt[4]{3 \times 3 \times 3 \times 3}$ to $\sqrt[4]{2 \times 2 \times 2 \times 2 \times 5}$, or 3 to $2\sqrt[4]{5}$; and conſequently the ratio of the vibrations of any conſonance tempered by a quarter of a comma, is alſo incommenſurable, as being compoſed of the ratio of the vibrations of the perfect conſonance, and the ratio of the ſingle vibrations which terminate its temperament.

The ſame may be ſaid of any perfect conſonance tempered by any aliquot part or parts of a comma; whoſe vibrations are always incommenſurable, becauſe 81 and 80 are not equal powers of any two numbers whatever (x). We may conclude then, that in tempered ſyſtems the vibrations of moſt of the conſonances are incommenſurable.

2. Now

(x) See coroll. Prop. 2. VIII. Elem. Euclid.

2. Now if the agreeable ſenſation of conſonances, according to the received principle in Harmonics, be the reſult of the frequent coincidences of their pulſes, and conſequently be more or leſs agreeable according as the coincidences are more or leſs frequent; all the conſonances in tempered ſyſtems, whoſe vibrations are incommenſurable, ought to be the greateſt diſcords in nature: it being impoſſible for their pulſes to coincide more than once in an infinite time. For as no two numbers how large ſoever, can expreſs the ratio of ſuch vibrations, ſo no multiple of one vibration can ever be equal to any multiple of the other. And yet experience ſhews that ſuch conſonances are much more agreeable than perfect diſcords whoſe pulſes coincide very often.

We may approach indeed as near as we pleaſe, and certainly much nearer than the ſenſe can diſtinguiſh, towards the exact magnitude of an incommenſurable ratio, by the ratios of whole numbers; but as theſe will grow larger and larger without bounds, ſo will the time between the ſucceſſive coincidences, or the length of the approximating cycle of the pulſes: by which I mean the time of either of the incommenſurable vibrations multiplied by the heterologous term of the approximating ratio.

Let any man tell us then where we may ſtop, and which of thoſe cycles it is, whoſe repetition excites the determinate ſenſation of the conſonance.

3. The like difficulty occurs in approaching gradually even to a commenſurable ratio of the vibrations of any perfect conſonance. For if either of its vibrations be pretty much altered at once, and then be made to approach by degrees to its former length, the terms of the ſeveral approximating ratios will grow larger and larger without bounds and in regular order, except when ratios occur whoſe terms are reducible; and the cycles of their pulſes will accordingly be longer and longer and their coincidences fewer and fewer without limit, thoſe interruptions excepted; and yet the conſonance will grow better and better by regular degrees till it arrives at perfection, as is certain by experience. For inſtance the ratios 30 to 21, 300 to 201, 3000 to 2001, &c, approach nearer and nearer to 3 to 2, and the v[ths] whoſe vibrations are in thoſe ratios grow more and more harmonious, though the cycles of their pulſes grow longer and longer to infinity.

4. It is therefore impoſſible to account for the phænomena of imperfect conſonances upon the principle of coincidences, which indeed is applicable to none but perfect ones. Accordingly Dr. *Wallis* (*y*), Mr.

(*y*) It hath been long ſince demonſtrated, that there is no ſuch thing as a juſt hemitone practicable in muſic, and the like for the diviſion of a tone into any number of equal parts, three, four or more. For ſuppoſing the proportion of a tone or full note to be as 9 to 8, that of the half note muſt be as $\surd 9$ to $\surd 8$, that is as 3 to $\surd 8$, or

Mr. *Euler* (z) and others diſapprove incommenſurable vibrations as impracticable and inharmonious.

5. But ſuppoſing the vibrations V, v of imperfect uniſons to be incommenſurable, or $V : v :: \sqrt{p} : \sqrt{q}$, and x to be an indeterminate vibration, and $V : x :: m : n$, and the ratios of the indeterminate numbers m, n to approach gradually to the given ratio of $\sqrt{p}$ to $\sqrt{q}$; though the length $n\,V, = m\,x$, of the indeterminate cycle of the pulſes of V and x, increaſes without bounds, nevertheleſs the length $\frac{n}{m-n}V, = \frac{m}{m-n}x$, of the indeterminate period of their pulſes tends gradually to a determinate limit $\frac{\sqrt{q}}{\sqrt{p}-\sqrt{q}}V = \frac{\sqrt{p}}{\sqrt{p}-\sqrt{q}}v$. And this is the period of the pulſes of the incommenſurable vibrations V, v, which excites the determinate ſenſation of the imperfect uniſons, be the complex cycle of their pulſes ever ſo long, infinite or impoſſible.

or as 3 to $2\sqrt{2}$, which are incommenſurable quantities; and that of a quarter note as $\sqrt[4]{9}$ to $\sqrt[4]{8}$, which is yet more incommenſurate; and the like for any other number of equal parts: which will therefore never fall in with the proportions of number to number. *Upon the imperfection of an Organ.* Phil. Tranſ. N°. 242, or Abridgm. vol. 1. p. 705. edit. 1.

(z) Denique ob nullam ſonorum rationem rationalem præter octavas, hoc genus [muſicum] harmoniæ maxime contrarium eſt cenſendum; etiamſi hebetiores aures diſcrepantiam vix percipiant. *Tentamen novæ Theoriæ muſicæ*, cap. IX. ſect. 17. Petropoli. 1739.

I ſay determinate ſenſation. For though the alternate leſſer intervals of the pulſes in the ſeveral ſucceſſive periods of V and v, even when commenſurate, are not preciſely equal (*a*), yet it is highly probable that the ear could not diſtinguiſh a repetition of any one period from the ſucceſſion of them all, and ſeems agreeable to experience in obſerving the identity of the tone of imperfect uniſons held out upon an organ.

6. For further illuſtration I will add an example or two. We ſhewed above that the vibrations V, v of the mean tone are as $\sqrt{5} : 2 :: 2.23606796$ &c $: 2 :: m : n$. Whence the length of the period of the pulſes of V and v, is $\frac{n}{m-n}\, V = \frac{2\, V}{0.23606796 \,\&c} = 8.47213 \,\&c \times V$; which is a medium between 8 V and 9 V, the cycles of the pulſes of the major and minor tones, ſomething leſs than the arithmetical, or even the geometrical mean, but not quite ſo little as the harmonical mean between them (*b*).

Again, when V and v are the vibrations of two ſounds whoſe interval is a quarter of a comma, we found $V : v :: 3 : 2\sqrt[4]{5}$ or 2.99069756 &c $:: m : n$; whence the period of the pulſes of V and v is $\frac{n}{m-n}\, V = \frac{2.99069756 \,\&c}{0.00930244 \,\&c} \times V = 321.4960 \,\&c \times V$.

Or

(*a*) Coroll. 2. Prop. VII.
(*b*) See Sect. VII. Def. II.

Or thus. In approximating towards the ratio of V to v, or 3 to $2\sqrt[4]{5}$, or 3 to 2.990697, or 3000000 to 2990697 by ſmall numbers (*c*), the ratios greater than V to v are 322 to 321, 967 to 964, 1612 to 1607, &c. Whence the cycle 321V and the periods $321\frac{1}{3}$V, $321\frac{2}{5}$V, &c, are all too ſhort.

And the ratios leſs than V to v being 323 to 322, 645 to 643, &c, the cycle 322V and periods $321\frac{1}{2}$V, &c, are all too long. Therefore the true period falls between the laſt mentioned limits, agreeably to the former computation.

From what has been ſaid of imperfect uniſons the difficulty vaniſhes in other imperfect conſonances, by obſerving the reduction of the periods of their imperfections to thoſe of imperfect uniſons, as in Prop. VIII.

7. *If the iſochronous vibrations of contiguous parcels of air, excited by different ſtrings, cannot be reduced to a ſynchroniſm by the mutual actions of the particles, (as I think they cannot,) it will follow that coincident pulſes are not neceſſary but only accidental to a perfect conſonance.*

For while an imperfect conſonance is ſounding, if the ratio of the vibrations be made perfect, as in tuning a muſical inſtrument, from the inſtant of this change the diſlocation of the pulſes, whatever it be, will continue unaltered

(*c*) See Mr. *Cotes*'s Harmonia Menſurarum, Prop. 1. Schol. 3.

tered in all the ſubſequent ſhort cycles; and thus the conſonance is perfect without any coincident pulſes, unleſs when the change of the ratio happens at the inſtant of the coincidence of two pulſes.

8. *This however ſeems indiſputable, that coincident pulſes are not neceſſary to ſuch harmony as the ear judges to be perfect.*

For if any long period of imperfect uniſons, intercepted between two beats, be lengthened greatly and indeterminately, as in tuning an inſtrument; any given part of it, as long as any muſical note, will approach indefinitely near to perfect uniſons; certainly nearer than the ear can diſtinguiſh, as being often doubtful of their perfection. And yet throughout that part (ſuppoſed to be ſmall in compariſon to the whole period) the pulſes of one ſound divide the intervals of the pulſes of the other very nearly in a given ratio, of any determinate quantity between infinitely great and infinitely ſmall, in proportion to the diſtance of that part from the periodical point or point of coincidence. Nevertheleſs the ear cannot diſtinguiſh any difference in the harmony of ſuch different parts, as is evident by often repeating the ſame conſonance, which can hardly begin conſtantly in the ſame place of the long period. And the ſame argument is applicable to any given conſonance, as being formed by intermitting a proper number of pulſes of each ſound of the imperfect uniſons: and the

the conclufion feems to be confirmed by the following experiment.

9. When any ftring of a violin or violoncello is moved by a gentle uniform bow, while its middle point being lightly touched by the finger, is kept at reft, but not preffed to the fingerboard; the two halves of the ftring will found perfect unifons, an eighth above the found of the whole; and will keep moving conftantly oppofite ways.

Becaufe the tenfion and ftiffnefs of the parts of the ftring on oppofite fides of the quiefcent point, compel them to oppofite and fynchronous motions, and thefe parts compel the next to the like motions, and fo on, to the ends of the ftring. Hence, becaufe thefe oppofite motions of the halves of the ftring communicate and propagate the like motions to the contiguous particles of air and thefe to the next fucceffively, it follows that different particles of air at the ear, placed any where in a perpendicular that bifects the whole ftring, will keep moving conftantly oppofite ways at the fame time; thofe particles, which received their motion from one half of the ftring, going towards the ear, while others are returning from it, which received an antecedent motion from the other half of the ftring: Or, in fewer words, the fucceffive pulfes of one found are conftantly bifecting the intervals between the pulfes of the other: And yet the harmony of the unifons is perfectly

perfectly agreeable to the ear, as I have often experienced.

10. And in ſo rare a fluid as air is, where the intervals of the particles are 8 or 9 times greater than their diameters (*d*), there ſeems to be room enough for ſuch oppoſite motions without impediment: eſpecially as we ſee the like motions are really performed in water, which in an equal ſpace contains 8 or 9 hundred times as many ſuch particles as air does (*d*). For when it rains upon ſtagnating water, the circular waves propagated from different centers, appear to interſect and paſs through or over each other, even in oppoſite directions, without any viſible alteration in their circular figure, and therefore without any ſenſible alteration of their motions.

11. If it be objected to the experiment above, that a conſtant biſection of the intervals of the pulſes of one of the uniſons by thoſe of the other, if true, ought to excite a ſenſation of a ſingle ſound an eighth higher than the uniſons, and as it does not, that of conſequence there is no biſection; a ſatisfactory anſwer to the objection might eaſily be drawn from the different duration and ſtrength of the ſingle pulſes of different ſounds at a different pitch, were it neceſſary to enter into that conſideration.

12. But

(*d*) Newt. Princip. Lib. 2. Prop. 50. Schol. and Prop. 23.

12. But after all, as abſolute certainty is difficult to be had in this inquiry, I choſe to give the vulgar definition of a perfect conſonance in Sect. III. Art. 3, as a ſimpler principle to build upon, and yet as fit for that purpoſe as a more general one would be, even ſuppoſing it were inconteſtable.

Scholium 5.

Having obſerved a very ſtrict analogy between the undulations of audible and viſible objects, I will here deſcribe it, as an illuſtration of the foregoing theory of imperfect conſonances.

Pl. XIII. Fig. 39. Let the points a, b, c, &c and α, β, γ, &c repreſent the places of two parallel rows of equidiſtant and parallel objects, ſuch as pales, palliſadoes, &c, and let them be viewed from any large diſtance by an eye at any point z. In a plane paſſing through the eye and cutting the axes of the parallel objects at right angles in the points, a, b, c, &c, α, β, γ, &c, let lines drawn from z through α, β, γ, &c, cut the line of the other row in A, B, C, &c. Then by the ſimilar triangles ABz and $\alpha\beta z$, BCz and $\beta\gamma z$, CDz and $\gamma\delta z$, &c, we have $AB : \alpha\beta ::$ $(Bz : \beta z ::)$ $BC : \beta\gamma ::$ $(Cz : \gamma z)$ $:: CD$ $: \gamma\delta ::$ &c. Therefore the antecedents AB, BC, CD, &c, which are to the equal conſequents $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, &c, in the ſame ratio,

are

are alſo equal to one another, and are the apparent projections of the conſequents upon the line $a\,b\,c$ of the other row.

Hence ſuppoſing m and n to repreſent the leaſt whole numbers in the given ratio of $A\,B$ to $a\,b$, we have a line $m \times a\,b = n \times A\,B$, equal to the length of the cycle between the apparent coincidences of ſome of the objects in one row with ſome in the other; as of α and a at A, of $\varkappa$ and m at K, &c: and if $m-n$ be not an unit we have a ſhorter line $\frac{m-n}{m}\,a\,b = \frac{n}{m-n}\,A\,B = A\,X$ or $X\,K$, equal to the length of the apparent period of their neareſt approaches towards coincidences; as on each ſide of the point X, according to the demonſtration of the VII^th^ propoſition.

But if the point z be ſo ſituated, that the lines $A\,B$ and $a\,b$ or $\alpha\,\beta$, or $B\,z$ and $\beta\,z$, or $C\,z$ and $\gamma\,z$, &c, which are all in the ſame ratio, happen to be incommenſurable, it will be impoſſible, mathematically ſpeaking, for more than one couple of objects to appear coincident (*e*), and yet the periods of their apparent approaches will ſubſiſt in this caſe as well as in the other.

Now if the objects be white, or of any colour that reflects more light to the eye than what comes to it from the ſpaces between them, and their breadth be conſiderable as uſual, the rows will appear the leaſt luminous about the coin-

(*e*) See Prop. XI. Schol. 4. Art. 2.

coincident objects and the periodical points, *A*, *X*, *K*, &c, where the objects of the nearer row hide the whole or some part of those behind them in the remoter row; and the rows will appear gradually more luminous towards the middle of the periods, where the objects will be seen distinct from one another if they be not too broad. And the contrary will happen if the objects in the rows be less luminous than the spaces between them.

Consequently if the spectator stands still and moves his eye from one end of the rows to the other, he will see an alternate succession of light and shade; and while he moves forwards in any transverse direction $z\,\omega$, and fixes his eye upon a given place of the rows, he will then see an undulation of light and shade, moving forwards quicker or slower according to the celerity of his own motion.

For then the apparent coincidences which were at *A*, *K*, &c, and consequently the intermediate periodical points *X*, *Y*, &c, will gradually shift from *A* to *B*, &c, and from *K* to *L*, &c, as is evident from the angular motion of the visual rays about the fixt points or objects α, β, γ, &c, κ, λ, μ, &c: And this is a known phænomenon.

If the spectator recedes from the rows, the period $\frac{m}{m-n}\,ab$ will grow longer, and upon his moving transversely, the visible undulations will be broader and slower than before, and at a very

very great distance from the rows, will become imperceptible; as being changed into an uniform appearance of both rows in the place of one: quite analogous to the audible undulations of imperfect unisons, as they grow slower and less perceptible while the unisons are approaching to perfection.

The like phænomenon results from two rows of pales that meet in any angle.

PROPOSITION XII.

Imperfect consonances of the same Name are equally harmonious when their short cycles are equally numerous in the periods of their imperfections.

As perfect consonances of the same Name are equally harmonious because their cycles are similarly divided by the pulses of their sounds; so imperfect consonances will be equally harmonious when their periods are similarly divided.

Hence all imperfect unisons whose single vibrations have the same ratio, are equally harmonious, as having similar periods (*f*); and therefore all imperfect consonances of the same name whose tempering ratios are the same, are equally harmonious.

For

(*f*) Prop. VII. Cor. 4.

For ſince the vibrations of the correſponding perfect conſonances have the ſame given ratio, *m* to *n*, and the vibrations of the imperfect ones are derived from thoſe of the ſimilar uniſons by intermitting $m-1$ and $n-1$ pulſes of their homologous vibrations, ſo as to leave equidiſtant pulſes in every ſeries (*g*); the ſimilar periods of the uniſons are thereby altered into ſimilar periods of imperfect conſonances; and the equal intervals of the uniſons into equal temperaments of the conſonances (*h*).

And the lengths of theſe ſimilar periods being proportional to the ſingle vibrations of their baſes or to equimultiples of them, that is, to the lengths of the ſhort cycles of the perfect conſonances, will contain equal numbers of imperfect ſhort cycles (*i*). Q. E. D.

Coroll. Conſonances of the ſame name are equally harmonious when equally and ſimilarly tempered.

Scholium.

After an organ had been well tuned by making all the tempered v^ths^ as equally harmonious as the ear could determine, I found that the numbers of their beats, made in equal times, were inverſely proportional to the times of

(*g*) See Dem. Prop. VIII. towards the end "And "Univerſally, &c.

(*h*) Prop. VIII. Cor. 1, 2.

(*i*) Ibid. Cor. 2.

of the single vibrations of their bases or trebles, as nearly as could be expected: or that the times between their successive beats, which are equal to the periods of their least imperfections (*k*), were directly proportional to those homologous vibrations, or to equimultiples of them, or to the lengths of the short cycles, which therefore were equally numerous in those periods.

PROPOSITION XIII.

Imperfect consonances of all sorts are equally harmonious, in their kind, when their short cycles are equally numerous in the periods of their imperfections.

Pl. XII. Fig. 34. The times of the single vibrations of imperfect unisons being represented by AB and ab, let AD and ac, that is $3AB$ and $2ab$ be those of imperfect v^ths^. And one length of their imperfect short cycle being $2AD = AG$, and the other being $3ac = ag$, their difference Gg is the dislocation of the pulses G, g at the end of the first short cycle $AagG$, measured from the coincident pulses Aa. And the greater of the two dislocations which terminate the several succeeding cycles, is double, triple, &c of Gg (*l*).

Again,

(*k*) Pro. X. (*l*) Prop. VII.

Again, conceiving the pulſes c, g, l, &c, to be now intermitted, let AD and ae, that is $3AB$ and $4ab$ be the ſingle vibrations of imperfect 4ths. And the two lengths of their firſt ſhort cycle $ANna$ being $4AD=AN$ and $3ae=an$, their difference Nn is the diſlocation of the pulſes N, n at the end of that cycle; and in the ſeveral ſucceeding cycles the greater of the two diſlocations is double, triple, &c of Nn.

And the common period AZ or $a\eta$ of thoſe diſlocations or imperfections in the ſhort cycles of the vths and 4ths, is the ſame as the period or ſimple cycle of the pulſes of the vibrations AB, ab of the imperfect uniſons (m).

Now the two diſlocations Gg, Nn, in the firſt imperfect cycles of the vths and 4ths in that period, are in the ratio of AG to AN (n), the lengths of the cycles, that is of $2AD$ to $4AD$, or 1 to 2: and the two greater diſlocations Xy, $\underset{\smile}{Q}r$, in the laſt imperfect cycles $Xy\varepsilon\Delta$, $\underset{\smile}{Q}r\varepsilon\Delta$, in the ſame period AZ, are in the ratio of their diſtances ZX, $Z\underset{\smile}{Q}$, from this end of it: and this ratio is leſs than that of ΔX to $\Delta \underset{\smile}{Q}$, or 1 to 2. But the two greater diſlocations $K\lambda$, $\Pi\sigma$ in the ſubſequent cycles $K\lambda\varepsilon\Delta$, $\Pi\sigma\varepsilon\Delta$, of the next period, are in the ratio of ZK to $Z\Pi$, which, on the contrary, is greater than that of ΔK to $\Delta\Pi$, or 1 to 2.

(m) Prop. VIII. (n) Prop. VII.

The periods muſt be conceived to contain a much greater number of ſhort cycles than can be well repreſented in a ſcheme. And then, as the correſponding diſlocations in the v^ths^ and 4^ths^ lie farther and farther from Z, the ratio of their diſtances and magnitudes will approach nearer and nearer to 1 to 2.

Therefore 1 to 2, or the ratio of the lengths of the ſhort cycles of the v^ths^ and 4^ths^, is either the exact or the mean ratio both of the greater and the leſſer diſlocations in all their correſponding ſhort cycles: becauſe the leſſer of the increaſing diſlocations in any ſubſequent cycle, is the ſame as the greater in the antecedent one.

Now while the length AG or ag remains unaltered, imagine the diſlocation Gg of the v^ths^ to be increaſed in that ratio of 1 to 2, and then it will be equal to the former magnitude of the diſlocation Nn of the 4^ths^, or to Nn in Fig. 35, ſuppoſing the pulſes C, G, L, &c to be abſent. And the firſt diſlocation Bb of the pulſes B, b, of the imperfect uniſons, being at the ſame time increaſed in the ſame ratio, their period AZ, which is alſo that of the diſlocations in the v^ths^ (*o*), will be diminiſhed very nearly in that ratio inverted (*p*). And thus the preſent period of the imperfect v^ths^ and the former period of the 4^ths^, are in the ratio of the lengths of their ſhort cycles; which therefore

(*o*) Prop. VIII (*p*) Cor. 7. Lemma to Prop. IX.

fore are equally numerous in their reſpective periods.

And ſince the greater and leſſer diſlocations at the ends of the correſponding ſubſequent ſhort cycles of the v^ths^ and 4^ths^, are now reſpectively equal, either exactly or at a medium of one with another, and equally numerous too, the whole periods compoſed of theſe ſhort cycles, will be equally harmonious. Becauſe thoſe equal diſlocations of the pulſes in the correſponding ſhort cycles, are the cauſes that ſpoil their harmony: and cauſes conſtantly equal will have equal effects.

The concluſion will be the ſame if the diſlocation Nn, in the firſt cycle of the 4^ths^ in either figure, be contracted to the magnitude of the diſlocation Gg belonging to the v^ths^ in the other. For then the new period of the 4^ths^, being double of the old one (*q*), will be to the old one, or that of the v^ths^, as AN to AG, that is, in the ratio of the lengths of their ſhort cycles, which therefore are equally numerous in theſe periods: and the diſlocations at the ends of the ſeveral ſubſequent ſhort cycles of the 4^ths^, being likewiſe contracted to the reſpective magnitudes of thoſe of the v^ths^, the conſonances are again made equally harmonious.

And laſtly, ſince either of thoſe conſonances is equally harmonious to another of the ſame name, at any other pitch, when their ſhort

(*q*) Cor. 7. Lemma to Prop. IX,

cycles are equally numerous in their periods (*r*), it appears that 4ths and vths are equally harmonious at any pitches, when their ſhort cycles are equally numerous in their periods. And the like proof is plainly applicable to any other caſe of theſe or any other conſonances: I mean when the common period of the imperfect uniſons is terminated at firſt either by coincident pulſes or periodical points; as will plainly appear by conceiving a ſhort cycle or two to reſult from a proper intermiſſion of the pulſes of imperfect uniſons on each ſide of ſuch points in fig. 24, 25. Q. E. D.

Coroll. 1. Imperfect conſonances are more harmonious in the ſame order as their ſhort cycles are more numerous in the periods of their imperfections.

For if any two imperfect conſonances be ſuppoſed equally harmonious, their ſhort cycles will be equally numerous in their periods, by the propoſition. Then if either of the given periods be lengthened, the ſhort cycles will be more numerous in it, and the leaſt diſlocation of their pulſes being ſmaller than before, and the greateſt much the ſame (*s*), the diſlocations will firſt increaſe and then decreaſe by ſmaller and more degrees from one end of the period to the other. And thus the conſonance will be more harmonious than it was at firſt, or than the other given conſonance.

And

(*r*) Prop. XII. (*s*) Prop. VII and VIII.

And on the contrary, if the period of either consonance be shortened, the number of its short cycles will be diminished, and the dislocations of their pulses will increase and decrease by larger and fewer degrees than before. And thus the consonance will be less harmonious than it was before, or than the other given consonance.

Coroll. 2. Imperfect consonances are more harmonious in the same order, as their temperaments multiplied by both the terms of the ratios of the single vibrations of the corresponding perfect consonances, are smaller; and are equally harmonious when those products are equal.

Pl. XII. Fig. 34, 35. For the vibrations of imperfect unisons being AB and ab and the terms of any perfect ratio of majority m and n, the vibrations of an imperfect consonance tempered sharp are mAB and nab, and those of the imperfect consonance tempered flat are mab and nAB; and the periods of the least imperfections in both have the same length as the period of the imperfect unisons (*t*); which length, supposing $AB : ab :: R : r$ in the least integers, is $\frac{r}{R-r} AB$; call it p.

Now the length of the imperfect short cycle of either of those imperfect consonances is $mnAB$ (*u*); call it c. Then

(*t*) Prop. VIII. (*u*) Prop. VIII. Cor. 2.

$\frac{p}{c} = \frac{r}{R-r} AB \times \frac{1}{mn\,AB} = \frac{1}{mn} \times \frac{r}{R-r} = \frac{1}{mnt}$ by taking $t = \frac{R-r}{r}$, which being as the logarithm of the tempering ratio $R : r$, or AB to ab (x) is very nearly as the temperament of both thofe confonances (y).

Therefore in the fame order in which the values of $\frac{p}{c}$ or $\frac{1}{mnt}$ are greater, or the values of mnt are fmaller, the correfponding confonances are more harmonious, by corol. 1; and are equally harmonious when the values of mnt are equal, by the prefent propofition.

Coroll. 3. Confequently imperfect confonances are equally harmonious when their temperaments have the inverfe ratio of the products of the terms of the perfect ratios of the correfponding perfect confonances.

For when the values of the product $mn \times t$ are equal, the values of t have the inverfe ratio of the values of mn.

Coroll. 4. When the products mn of the terms of the perfect ratios are equal, the tempered confonances are more harmonious in the fame order as their temperaments are fmaller; and are equally harmonious if their temperaments be equal.

For

(x) Cor. 2. Lemma to Prop. IX and Prop. VIII cor. 1.
(y) Sect. 1. Art. 11.

For if the values of mn or $\frac{1}{mn}$ be equal, the values of $\frac{p}{c} = \frac{1}{mn} \times \frac{1}{t}$ are greater in the ſame order as thoſe of $\frac{1}{t}$ are greater, or as thoſe of t are ſmaller; and are equal when the values of t are equal.

Coroll. 5. Therefore imperfect conſonances of the ſame Name are more harmonious in the ſame order as their temperaments are ſmaller; and are equally harmonious when they are equal.

Becauſe the terms of the perfect ratios of conſonances of the ſame name are the ſame, and their product the ſame.

Coroll. 6. Imperfect conſonances equally tempered are more harmonious in the ſame order as the products of the terms of the perfect ratios belonging to the perfect conſonances are ſmaller; and are equally harmonious when thoſe products are equal.

For the values of t being ſuppoſed equal, thoſe of $\frac{p}{c} = \frac{1}{mn} \times \frac{1}{t}$ are greater in the ſame order as the values of $\frac{1}{mn}$ are greater, or as thoſe of mn are ſmaller; and the former values are equal when the latter are ſo.

Coroll. 7. Imperfect conſonances equally tempered are generally more harmonious in the ſame order as they are ſimpler, the pure ones chiefly excepted (z), which are more harmo-

(z) Sect. III. Art. 8.

nious than ſome others that are ſimpler; though ſeparately conſidered they follow that order exactly.

This will appear from the ſixth corollary by a ſeries of the products of the terms of the ratios in the firſt column, compared with the ſeries of numbers in the ſecond column of the table in Sect. III. Art. 5, ſhewing the order of the ſimplicity of conſonances.

Coroll. 8. Conſequently ſimpler conſonances will generally bear greater temperaments than the leſs ſimple will; or the leſs ſimple ones generally ſpeaking will not bear ſo great temperaments as the ſimpler will: contrary to the common opinion (*b*).

Coroll. 9. The tempered concords in the ſyſtem of mean tones (*c*) are not equally harmonious in their kinds.

For by Coroll. 6, and by inſpection of the terms of the perfect ratios annexed to the characters of the concords in the firſt of the tables in

(*b*) Octavæ autem fiant exactæ; nam vel minimus octavæ defectus fit intolerabilis. *Dechales* Curſus math. Tom. IV. de Muſica, cap. XI.

(*b*) Octavarum autem omnium unica eſt ſpecies, eaque perfecta ratione 1 ad 2 contenta. Hoc enim intervallum, propter perfectionem, vix aberrationem à ratione 1 ad 2 pati poſſet, quin ſimul auditus ingenti moleſtia afficeretur. Namque quo perfectius perceptuque facilius eſt intervallum, eo magis ſenſibilis fit error minimus; minus autem ſentitur exiqua aberratio in intervallis minus perfectis. *Tentamen novæ Theoriæ muſicæ*, cap. IX. ſect. 10. *Petropoli* 1739.

(*c*) Prop. 2.

in the next ſection, it will appear, that the v^th^ and 4^th^ and their compounds with VIII^ths^, are more harmonious than the VI^th^ and 3^d^ and their compounds with equal numbers of VIII^ths^, as being all equally tempered in that ſyſtem (*d*).

Coroll. 10. The harmony of thoſe concords is ſtill more unequal in the *Hugenian* ſyſtem, reſulting from a diviſion of the octave into 31 equal intervals (*e*).

Becauſe the common temperament of the VI^th^ and 3^d^ and their compounds with VIII^ths^, which by Coroll. 4 and 9, ſhould be ſmaller than that of the v^th^ and 4^th^ and their compounds with VIII^ths^, to render them equally harmonious, is on the contrary ſomething greater.

Coroll. 11. Imperfect conſonances are more harmonious both as they beat ſlower, and as the cycles of the perfect conſonances are ſhorter.

For the quantities $\frac{p}{c}$ will be greater on both accounts (*f*) and the harmony better (*g*).

Coroll. 12. Imperfect conſonances having the ſame Baſe are more harmonious in the ſame order as their Beats made in equal times and multiplied by the Minor terms of the perfect ratios of the reſpective perfect conſonances are ſmaller: and are equally harmonious when thoſe products are equal, that is, when the beats are inverſely as the minor terms (*h*).

For

(*d*) Prop. III. Coroll. 3.
(*e*) See Prop. XVII, Scholium.
(*f*) Prop. XI.
(*g*) Prop. XII and XIII.

For let the ſingle vibrations of the baſe and treble of an indeterminate perfect conſonance be Z and z, and $Z : z :: m : n$ in the leaſt numbers, then the ſhort cycle $c = nZ = mz$, and putting β for the number of beats made in any given time by the correſponding imperfect conſonance, the period p is as $\frac{1}{\beta}$ as being equal to the interval of the ſucceſſive beats (*i*); and the harmony being as $\frac{p}{c}$ or $\frac{1}{\beta n Z}$ or $\frac{1}{\beta m z}$, is better as the values of βn are ſmaller if Z be conſtant, or as βm is ſmaller, if z be conſtant, by Coroll. 1. Prop. XII.

Coroll. 13. Hence if the Baſes and Beats be the ſame, the harmony is better as the minor terms are ſmaller and equally good when they are the ſame: or if the Baſes and Minor terms be the ſame, it is better as the beats are ſlower, and equally good when they are iſochronous.

Coroll. 14. And the two laſt corollaries are applicable to trebles and major term, by reading trebles inſtead of baſes and major terms inſtead of minor, as appears by the demonſtration.

(*h*) See Pl. 1. Fig. 3. and Plate XII. Tab. 1.
(*i*) Prop. X.

SECTION

SECTION VII.

Of a ſyſtem of ſounds wherein as many concords as poſſible, at a medium of one with another, ſhall be equally and the moſt harmonious.

DEFINITION I.

The Arithmetical Mean among any number of quantities, is to the ſum of them under their given ſigns, as an unit is to their number; and has the ſame ſign as their ſum has: Or if they be expreſſed by numbers, it is the quotient of their ſum divided by their number.

Thus the arithmetical mean among the quantities a, b, c, $-d$, is $\frac{a+b+c-d}{4}$.

Coroll. 1. Hence the ſum of the exceſſes of all the greater quantities above their arithmetical mean, is equal to the ſum of the defects of all the ſmaller from the ſame.

For let the arithmetical mean $\frac{a+b+c-d}{4}$ $=r$, then $a+b+c-d=4r=r+r+r+r$.

Whence

Whence if a and b be ſeverally greater than r, we have $a—r+b—r=r—c+r+d$.

Pl. XIII. Fig. 40. Accordingly if the lines ao, bo, $co—do$ and ro, be proportional to $a, b, c, —d$ and r, the ſum of the parts ra, rb on one ſide of the point r, is equal to the ſum of the parts rc, rd on the other ſide of it.

Coroll. 2. If any quantity q be added to, or taken from every one of the quantities $a, b, c, —d$, their arithmetical mean will accordingly be augmented or diminiſhed by that quantity q.

For let $a+b+c—d=4r$, then r is their arithmetical mean. But $a+q+b+q+c+q—d+q=4r+4q=4\times\overline{r+q}$, and therefore $r+q$ is the arithmetical mean among thoſe augmented quantities $a+q$, $b+q$, $c+q$, $—d+q$: and by changing the ſign of every q, it appears that $r—q$ is the like mean among the diminiſhed quantities $a—q$, $b—q$, $c—q$, $—d—q$.

Coroll. 3. If every one of the quantities $a, b, c, —d$, be increaſed or diminiſhed in any given ratio of 1 to n, their arithmetical mean will alſo be increaſed or diminiſhed in the ſame given ratio.

For let $a+b+c—d=4r$, then r is their arithmetical mean. But $na+nb+nc—nd=4nr$, and therefore nr is the arithmetical mean among the quantities na, nb, nc, $—nd$.

DE-

DEFINITION II.

The Harmonical Mean among any number of quantities, is the reciprocal of the arithmetical mean among their reciprocals.

For inſtance, the reciprocals of a, b, c, are $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, whoſe arithmetical mean is $\frac{1}{3} \times \overline{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$ and its reciprocal $\dfrac{1}{\frac{1}{3} \times \overline{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}}$ is the harmonical mean among a, b, c; where 1 ſignifies any given conſtant quantity.

Pl. XIII. Fig. 41. Likewiſe in any hyperbola where the ordinates parallel to an aſymptote are the reciprocals of their abſciſſes, meaſured from the center upon the other aſymptote; if an abſciſs ro be the arithmetical mean among the abſciſſes ao, bo, co, do, its ordinate $r\rho$ is the harmonical mean among the ordinates $a\alpha$, $b\beta$, $c\gamma$, $d\delta$ by the definition, $r\rho$ being the reciprocal of the arithmetical mean ro among their reciprocals, ao, bo, co, do.

Likewiſe if an ordinate $m\mu$ be the arithmetical mean among the ordinates $a\alpha$, $b\beta$, $c\gamma$, $d\delta$, its abſciſs mo is the harmonical mean among their abſciſſes, ao, bo, co, do: and on the contrary.

Coroll.

Coroll. 1. The arithmetical mean is greater than the harmonical mean among the same quantities, if they all have the same sign.

For let the line $\beta\varrho$, produced through the top of either of the ordinates next to $r\varrho$, cut the rest in f, g, h, and the asymptote ro in e. Then because ro is the arithmetical mean among ao, bo, co, do, the line re is the arithmetical mean among the lines ae, be, ce, de (*k*); and $r\varrho$ the arithmetical mean among the proportional lines af, $b\beta$, cg, dh (*l*), which, excepting the common ordinate $b\beta$, are severally smaller than the hyperbolical ordinates, $a\alpha$, $b\beta$, $c\gamma$, $d\delta$; whose arithmetical mean $m\mu$ is therefore greater than $r\varrho$ (*m*), the harmonical mean among the same ordinates.

Coroll. 2. The difference between the arithmetical and the harmonical means among the same quantities, will be very small when the differences of the quantities themselves are so.

This will appear by conceiving the ordinates $a\alpha$, $b\beta$, $c\gamma$, $d\delta$ to approach gradually towards one another till they coincide. For then the differences between the hyperbolical ordinates $a\alpha$, $b\beta$, $c\gamma$, $d\delta$, and the lines af, $b\beta$, cg, dh, and consequently between their arithmetical means $m\mu$, $r\varrho$, will gradually decrease to nothing. But $r\varrho$ is also the harmonical mean among those ordinates.

Coroll.

(*k*) Defin. 1. Coroll 1 or 2.
(*l*) Defin. 1. Coroll. 3. (*m*) Defin. 1.

Coroll. 3. By increaſing every quantity in any given ratio, the harmonical mean among them will be increaſed in the ſame ratio.

For the reciprocals of the increaſed quantities, and the arithmetical mean among them (*n*), will ſeverally be diminiſhed in that ratio; and the reciprocal of this mean, which is the harmonical mean among the increaſed quantities, will of conſequence be increaſed in the ſame ratio.

Coroll. 4. Fig. 42, 43. Whatever be the ſigns of the propoſed quantities $a\alpha$, $b\beta$, $c\gamma$, $d\delta$, their harmonical mean $r\rho$ has always the ſign of the ſum of their reciprocals ao, bo, co, do, or of ro, the arithmetical mean among them.

For the reciprocal of each quantity has the ſign of the quantity itſelf, and according as their ſum is affirmative or negative, ſo is their arithmetical mean (*o*), and ſo is its reciprocal, or the harmonical mean among the propoſed quantities.

(*n*) Defin. 1. Coroll. 3. (*o*) Defin. 1.

PRO-

PROPOSITION XIV.

Instead of several imperfect concords differently tempered and belonging to the same perfect one, if it be necessary to use but one, let the period of its imperfections be the arithmetical mean among all the periods of those concords, and it will best answer the several purposes of every one.

Because the excesses of the longer periods above the arithmetical mean are equal, one with another, to the defects of the shorter from the same, and because the arithmetical mean period is longer (*p*) and therefore more harmonious (*q*) than the harmonical mean among the same. Q. E. D.

(*p*) Def. 2. Coroll. I. Sect. VII.
(*q*) Prop. XIII. Coroll. I.

PRO-

PROPOSITION XV.

The tempered concords in any one of the parcels derived from the V[th], VI[th] *or* III[d] (*r*), *whatever be their common temperament, are constantly more harmonious in the same immutable order as the products of the terms of the perfect ratios belonging to the respective perfect concords are smaller; and those concords only are equally harmonious which have equal products belonging to them; and no others can be made so, because they cannot have different temperaments while the octaves are perfect.*

The truth of this proposition appears from prop. XIII. coroll. 6. and prop. III. Q. E. D.

(*r*) In the scholium to prop. III the concords were distributed into three parcels, which may be seen in one view in Table I placed after schol. 2. prop. XVI.

PROPOSITION XVI.

To find the temperament of a system of sounds of a given extent, wherein as many concords as possible, at a medium of one with another, shall be equally and the most harmonious.

Into the second and third columns of the II[d] Table following (*s*), transfer every couple of concords in the given system, whose characters can be taken from the different parcels in the I[st] Table; omitting all other couples whose characters are both situated in any one of the parcels. And after each couple place the ratio of the temperaments which would make the two concords equally harmonious in their kind (*t*).

Then will the corollaries to the IV[th], V[th] and VI[th] propositions give the temperaments themselves, or the positions of the temperers *O r s t*, *O r s t*, &c, in Fig. 44, Pl. XIV, belonging to every one of those ratios.

Among the temperaments in the three several parcels *G r*, *G r*, &c, *A s*, *A s*, &c, *E t*, *E t*, &c, taking three harmonical means, *GD*, *AH*, *EM*, and transferring them to Fig. 45, draw three temperers *O D e f*, *O g H i*, *O k l M*, and taking

(*s*) After Schol. 2. of this prop.
(*t*) Prop. XIII. coroll. 3.

taking $E q$ the arithmetical mean among the three temperaments $E M$, $E f$, $E i$, the temperer $O n p q$ will approach very near to the position required in the propofition.

If greater exactnefs be defired, among the three temperaments in the three feveral parcels $G D$, $G g$, $G k$; $A H$, $A e$, $A l$; $E M$, $E f$, $E i$, taking three harmonical means, $G D'$, $A H'$, $E M'$, and transferring them to Fig. 46, draw three new temperers $O D' e' f'$, $O g' H' i'$, $O k' l' M'$; and taking $E q'$ the arithmetical mean among the three temperaments $E M'$, $E f'$, $E i'$, the temperer $G n' p' q'$ will approach ftill nearer towards the required pofition.

And by repeating the like conftruction we may approach as near as we pleafe (u). Q. E. I.

THE DEMONSTRATION.

In combining the concords all the couples whofe characters are both in any one of the parcels in Tab. 1 are omitted; their harmony with refpect to one another, or the proportions of their periods being immutable, by reafon of their common temperament (x).

Pl. XIV. Fig. 44. Now fuppofing $G a$ to be the arithmetical mean among all the temperaments

I 2 $G r$,

(u) Want of room in fuch fmall plates made it neceffary to alter the true proportions of the lines in the figures; otherwife fome parts of them would have appeared confufed.

(x) Prop. XV.

Gr, *Gr*, &c, of the firft parcel of concords, and drawing the temperer *Oabc*, the temperaments *Ab* and *Ec* will be the like Means among *As*, *As*, &c, and *Et*, *Et*, &c, (*y*). Whence if the periods of concords of the fame name to the fame bafe were proportional to their temperaments, the beft temperer would be *Oabc* (*z*):

Becaufe the period of the v[th], for inftance, belonging to the temperament *Ga*, would then be the arithmetical mean among all the other periods of the v[ths] to the fame bafe, anfwering to the feveral temperaments *Gr*, *Gr*, &c. And the like may be faid of the periods of the 4[th] and of every other concord in this firft parcel, as having the temperaments *Gr*, *Gr*, &c, common to them all, and likewife of the periods of the feveral concords in the other two parcels, with refpect to their temperaments *Ab*, *As*, &c, and *Ec*, *Et*, &c.

But fince the periods of concords of the fame name to the fame bafe are (not directly but) inverfely proportional to their temperaments (*a*); the period of the v[th], or any other concord, belonging to the arithmetical mean temperament *Ga* is (not the arithmetical but) the harmonical mean among the other periods of that name, anfwering to the temperaments *Gr*, *Gr*, &c; and confequently is fhorter (*b*) and therefore

(*y*) Def. 1. coroll. 1. 2. 3. Sect. VII.
(*z*) Prop. XIV.
(*a*) Prop. IX. coroll. 4.
(*b*) Def. 2. coroll. 1. fect. VII.

fore leſs harmonious than the arithmetical mean period among them (*c*) anſwering to the harmonical mean temperament *G D*: And what has been ſaid of the periods in that parcel is applicable to thoſe of the other two, with reſpect to the arithmetical and harmonical mean temperaments *Ab* and *A H*, *Ec* and *E M*.

Therefore the arithmetical mean periods belonging to the harmonical mean temperaments *G D*, *A H*, *E M*, would beſt anſwer the deſign of the propoſition, if the points *D*, *H*, *M* were all ſituated in one temperer.

For ſince the ſums of the temperaments terminated at the ſeveral temperers *Orst*, *Orst*, &c, are the leaſt that can render the concords in each couple equally harmonious in their kind (*d*), it follows that the ſums of all the temperaments *G r*, *G r*, &c, in the firſt parcel, of all the temperaments *A s*, *A s*, &c, in the ſecond, and of all the temperaments *E t*, *E t*, &c, in the third, taking one ſum with another, are alſo the leaſt poſſible: the ſum total being the ſame in both diſtributions of the particulars.

The ſum of the harmonical temperaments *G D*, *A H*, *E M* being therefore the leaſt poſſible (*e*), and that of all the corresponding arithmetical mean periods being the greateſt (*f*), would render the ſyſtem of periods at a me-

(*c*) Prop. XIII. coroll. I.
(*d*) Prop. IV. V. VI.
(*e*) Def. I, and cor. I. Def. 2. ſect. VII.
(*f*) Prop. IX. cor. 4.

dium of one with another, the most harmonious.

But in reality the three harmonical points D, H, M cannot fall into any one temperer. For the concords in the first parcel being simpler than those in the second and third (*g*) and therefore requiring a smaller temperament (*h*), it appears, by cor. 6, 7, 8, prop. III, that the best temperer of the system must lie within the angle AOE, and so must the arithmetical mean temperer $Oabc$, as lying not far from the best; and therefore must have the points G, E on one side of it and A on the other: And the harmonical means GD, EM, AH being less than the respective arithmetical means Ga, Ec, Ab (*i*), the points D and M must lie on the same side of $Oabc$ as G and E do, and H on the other side. Therefore if a temperer could pass thro' D and M, yet it could not pass thro' H.

Pl. XV. Fig. 45. In the solution of the problem it was therefore necessary to reduce the three temperers $ODef$, $OgHi$, $OklM$ to one, by so drawing the temperer $Onpq$, as to make Eq the arithmetical mean among EM, Ef, Ei, and consequently Ap the like mean among AH, Ae, Al, and Gn the like mean among GD, Gg, Gk (*k*).

Now the differences of the three temperaments in each of those parcels being but small, as

(*g*) Tab. I following, compared with art. 5. sect. III.
(*h*) Prop. XIII. cor. 8.
(*i*) Defin. 2. cor. I. sect. VII.
(*k*) Def. I. coroll. I. 2. 3.

as will appear by the following calculation (*l*), the arithmetical means among them will differ but little from the reſpective harmonical means among the ſame (*m*), which would be fitter for the purpoſe if their extremities D', H', M' could be ſituated in any one temperer (*n*). Conſequently as the temperer $Onpq$ falls in the middle among the three temperers conceived to paſs through the harmonical points D', H', M', it will nearly anſwer the ſeveral purpoſes of thoſe three, and approach very near to the ſituation of the required temperer.

Pl. XVI. Fig. 46. Hence and by prop. XIV, it appears that a repetition of this laſt conſtruction, as deſcribed in the ſolution, will give a temperer $O\,n'\,p'\,q'$ approaching ſtill nearer to the required ſituation. Becauſe the latter temperaments EM', Ef', Ei' differ leſs from one another (*o*) and conſequently from their arithmetical mean Eq', than the former, EM, Ef, Ei, did from one another and from their arithmetical mean Eq.

And as the ſame may be ſaid of the temperaments of the other two parcels, it appears that by a further repetition of the ſame conſtruction, we may find a temperer approaching as near as we pleaſe towards the poſition required in the propoſition. Q. E. D.

Coroll. 1. Fig. 45. The comma, or four times the line $G\,1$, being the unit, and ſuppoſing any

I 4 three

(*l*) Tab. VI. column 2.
(*m*) Def. 2. coroll. 2. (*n*) Prop. XIV.
(*o*) Tab. VII. at the end of it.

three temperaments of different parcels to be given, as $GD=d$, $AH=b$ and $EM=m$, it will be easy to collect, (from the similar triangles under the line $O\,1\,3\,E$, the three temperers $ODcf$, $OgHi$, $OklM$, and the three parallels GD, AH, EM,) that $Gg=\frac{1-b}{3}$ and $Gk=\frac{1+m}{4}$; $Ae=1-3d$ and $Al=\frac{1-3m}{4}$; $Ef=4d-1$ and $Ei=\frac{1-4b}{3}$; provided the three temperers be all situated within the angle EOA; but if OH or OM lies out of it beyond A or E respectively, the sign of b or m will accordingly be changed in those theorems.

Coroll. 2. Hence we have the three arithmetical mean temperaments, $Gn=\frac{1}{3}\times\overline{d+\frac{1-b}{3}+\frac{1+m}{4}}$, $Ap=\frac{1}{3}\times\overline{1-3d+b+\frac{1-3m}{4}}$, and $Eq=\frac{1}{3}\times\overline{4d-1+\frac{1-4b}{3}+m}$.

Scholium 1.

Pl. XVII. Fig. 47 serves to illustrate part of the demonstration of the proposition, by representing to the eye the proportions of the periods of the concords. It is thus constructed. The line AI being parallel to EO, the middlemost parcel of hyperbolas $v\frac{1}{5}$, $x\frac{1}{6}$, $y\frac{1}{10}$, $z\frac{1}{12}$, &c, are drawn to the asymptotes AI, $A3$; and their ordinates to their common abscis $A3$ are made proportional to the fractions $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{10}$, $\frac{1}{12}$, &c.

Hence

Hence when the imperfect VI, 3^d, VI + VIII, 3^d + VIII, &c, to the ſame baſe are each tempered by a quarter of a comma, repreſented by the common abſciſs $A3$, their periods are proportional to thoſe ordinates (p); and when they have any other temperament repreſented by the abſciſs As, their periods are then proportional to the ordinates sv, sx, sy, sz, &c. (q).

And the like conſtruction being made for the other two parcels of concords, the ordinates erected from the interſections r, s, t of any temperer $Orst$, ſhew the proportions of the periods in the whole ſyſtem: and theſe proportions are the ſame whatever be the unit of the fractional ordinates.

Scholium 2.

In order to calculate the required temperaments of a ſyſtem of any given extent, it will not be amiſs to explain the following tables.

1. According to the ſolution of the problem, ſee whether every two characters of the concords, each of which lie in the different parcels in Tab. I, be placed over againſt one another in the ſecond and third columns of Tab. II^d. Part I.

2. Then examine whether the ratios placed after thoſe characters in the 4^{th} column of that table, be rightly deduced from the fractions annexed

(p) Prop. IX. coroll. 5. and Tab. I. at the end of the next Scholium.

(q) Prop. IX. coroll. 4.

nexed to the same characters in Tab. I, according to the rule in prop. XIII. coroll. 3.

3. See whether all the temperaments in Tab. IV be rightly deduced from those ratios in Tab. II[d], by the corollaries to prop. IV, V, VI; and whether the numbers in the first column of each table correspond to the same ratios and concords.

4. Examine whether the reciprocals in Tab. V, of the temperaments in Tab. IV be right, that is, whether the product of the quotient by the divisor, differs from the dividend by less than half the divisor. When a reciprocal is negative, as coming from a negative temperament of the III[d] or VI[th], which lies wholly out of the angle AOE, I subtract it from 0 and place the remainder in the table instead of the reciprocal itself. Thus at N° 10, —6) 26 (= —4.33333 &c, which subtracted from 0 gives $\bar{5}$.66667 to be transferred to Tab. II[d] Part II[d] and there added to the positive reciprocals, for the sake of uniformity in the work; the integer $\bar{5}$ being only negative and the decimals .66667 affirmative. For n being any given integer, the number $n - 4 - \frac{1}{3} = n - 5 + \frac{2}{3}$.

5. See whether the reciprocals in Tab. V be rightly transferred into the respective columns of Tab. II[d] Part II[d], which is readily done by means of the corresponding numbers in the first column of each table.

6. Cast up the several dozens of reciprocals in Tab. II[d] Part II[d], and transfer the sums to Tab. III and there cast them up.

7. Tab.

7. Tab. VI is thus deduced from Tab. III. By the ſolution of the problem the fraction $\frac{12}{42.72013}$, $= GD$ in Fig. 44, is the harmonical mean among the temperaments Gr, Gr, &c; becauſe its reciprocal $\frac{42.72013}{12}$ is the arithmetical mean among their reciprocals, as being their ſum divided by their number. The ſame is to be underſtood of all the other fractions: and as the value of the temperament Eq, computed from coroll. 2. prop. XVI, comes out affirmative, by the coroll. to prop. IV, V, VI, it is part of the interval EC of the perfect III^d, and therefore is a negative temperament of that concord, or an affirmative one of its complement to the octave. This is the firſt approximation towards the required temperament.

8. Tab. VII contains the calculation of Eq', the ſecond approximation towards the true temperament of the III^d, in a ſyſtem whoſe extent is but one octave, and is ſufficiently evident from cor. 1, 2, prop. XVI, and Tab. VI. And by a like calculation the values of Eq', in a ſyſtem of two and of three octaves, will be found as put down under thoſe of Eq in Tab. VI; care being taken in the operations to continue the quotients in decimals as far as they are juſt.

9. Therefore the reſult of the whole is this. As all the parts of muſical compoſitions in any given place (ſetting aſide double baſes) are generally contained within three octaves, and as their harmony is ſtronger and better within that

compaſs

compaſs than it would be in a larger; I chuſe to make all the concords within every three octaves equally harmonious and no more, be the extent of the ſyſtem ever ſo great; and conſequently to diminiſh the IIId by $\frac{1}{9}$ comma, this being very nearly the value of the laſt $Eq' = 0.11024$ in Tab. VI.

10. Hence in the ſyſtem of equal harmony the temperaments of the Vth, VIth and IIId are $\frac{-5}{18}$, $\frac{+3}{18}$ and $\frac{-2}{18}$ of a comma reſpectively (*r*) and are proportional to the muſical primes 5, 3 and 2. (*s*)

11. In determining theſe temperaments of the diatonic ſyſtem, I have regarded no more conſonances than the concords. 1. Becauſe the diſcords are ſeldomer uſed than the concords. 2. Becauſe the ear is generally leſs critical in the diſcords than in the concords. 3. Becauſe a mean temperament among thoſe of the concords and diſcords too, would differ from that of the concords alone, and therefore be leſs ſuitable to them.

12. Laſtly I have kept the octave perfect. 1. Becauſe it is the ſimpleſt and moſt harmonious

(*r*) Prop. III. and its 2d and 3d coroll.

(*s*) But if any one chuſes to have all the concords in 4 octaves made equally harmonious, he will find by continuing the tables, that the IIId muſt be diminiſhed by $\frac{987}{10000}$ of a comma, which being nearly $\frac{1}{10}$ comma, the temperaments of the Vth, VIth and IIId will then be $\frac{-11}{40}$, $\frac{+7}{40}$ and $\frac{-4}{40}$ of a comma reſpectively.

nious of all the concords, both in itſelf and its multiples. 2. Becauſe ſome one interval muſt be kept perfect, in order to determine the variations of the temperaments of the reſt (*t*). 3. Becauſe upon ſeveral trials of keeping other intervals perfect inſtead of the octave, many reaſons have occurred to me for rejecting every one of them.

13. Does it not follow then, that the ſyſtem of equal harmony, as above derived from the beſt ſyſtem of perfect intervals (*u*), is the beſt tempered and moſt harmonious ſyſtem that the nature of ſounds is capable of? (*x*).

14. It may not be amiſs to obſerve that in Fig. 44, 45, 46, $Ec - Eq$, the difference of the Arithmetical and Harmonical mean temperaments of the IIId, computed for one octave is $\frac{1}{214}$, for two is $\frac{-1}{98}$, for three is $\frac{-1}{69}$ of a comma. Hence in 3 octaves the arithmetical and harmonical mean temperaments of the vths are as 76 to 77 very nearly, and if the baſes of any vth in each ſyſtem be uniſons, their beats made in equal times are alſo as 76 to 77 (*y*): whence I judge that the harmony of the ſounds in the two ſyſtems can ſcarce be ſenſibly different (*z*). Nevertheleſs it appears by the demonſtration of the propoſition, that an accurate ſolution of it required the help of Harmonical Means.

(*t*) Prop. III. (*u*) Sect. IV. Art. 7.
(*x*) See Scholium. Prop. III. (*y*) Prop. XI. cor. 4.
(*z*) Prop. XI. ſchol. I. art. 4.

TAB.

TAB. II. PART I.

N°	Ratios of the temperaments for equal harmony of the		
1	V and	VI as	5 : 2
2	V	3^{d}	5 : 1
3	4^{th}	VI	5 : 4
1	4^{th}	3^{d}	5 : 2
4	V	III	10 : 3
5	V	6^{th}	20 : 3
1	4^{th}	III	5 : 3
4	4^{th}	6^{th}	10 : 3
6	VI	III	4 : 3
7	VI	6^{th}	8 : 3
1	3^{d}	III	2 : 3
6	3^{d}	6^{th}	4 : 3
	In one	Octave	1 dozen
2	V and	VI + VIII	5 : 1
8	V	3^{d} + VIII	10 : 1
1	4^{th}	VI + VIII	5 : 2
2	4^{th}	3^{d} + VIII	5 : 1
1	V	III + VIII	5 : 3
9	V	6^{th} + VIII	40 : 3
10	4^{th}	III + VIII	5 : 6
5	4^{th}	6^{th} + VIII	20 : 3
1	VI	III + VIII	2 : 3
11	VI	6^{th} + VIII	16 : 3
12	3^{d}	III + VIII	1 : 3
7	3^{d}	6^{th} + VIII	8 : 3
			2 dozen

TABLE I. facing p. 143.

Contains the characters and terms of the perfect ratios of all the concords.

1st Parcel.	2d Parcel.	3d Parcel.
V. $\frac{2}{3}$	VI. $\frac{3}{5}$	III. $\frac{4}{5}$
4th. $\frac{3}{4}$	3d. $\frac{5}{6}$	6th. $\frac{5}{8}$
V + VIII. $\frac{1}{3}$	VI + VIII. $\frac{3}{10}$	III + VIII. $\frac{2}{5}$
4th + VIII. $\frac{3}{8}$	3d + VIII. $\frac{5}{12}$	6th + VIII. $\frac{5}{16}$
V + 2VIII. $\frac{1}{6}$	VI + 2VIII. $\frac{3}{20}$	III + 2VIII. $\frac{1}{5}$
4th + 2VIII. $\frac{3}{16}$	3d + 2VIII. $\frac{5}{24}$	6th + 2VIII. $\frac{5}{32}$
V + 3VIII. $\frac{1}{12}$	VI + 3VIII. $\frac{3}{40}$	III + 3VIII. $\frac{1}{10}$
4th + 3VIII. $\frac{3}{32}$	3d + 3VIII. $\frac{5}{48}$	6th + 3VIII. $\frac{5}{64}$
&c.	&c.	&c.
&c.	&c.	&c.

TAB. II. PART II.

N°	Reciprocals of the temperaments of the		
	V, 4th & Comp.	VI. 3d & Comp.	III, 6th & Comp.
1	3.40000	8.50000	5.66667
2	3.20000	16.00000	4.00000
3	3.80000	4.75000	19.00000
1	3.40000	8.50000	5.66667
4	3.70000	5.28571	12.33333
5	3.85000	4.52941	25.66667
1	3.40000	8.50000	5.66667
4	3.70000	5.28571	12.33333
6	3.57143	6.25000	8.33333
7	3.72727	5.12500	13.66667
1	3.40000	8.50000	5.66667
6	3.57143	6.25000	8.33333
Sums	42.72013	87.47583	126.33334
2	3.20000	16.00000	4.00000
8	3.10000	31.00000	3.44444
1	3.40000	8.50000	5.66667
2	3.20000	16.00000	4.00000
1	3.40000	8.50000	5.66667
9	3.92500	4.24324	52.33333
10	5.20000	2.36364	5.66667
5	3.85000	4.52941	25.66667
1	3.40000	8.50000	5.66667
11	3.84211	4.56250	24.33333
12	3.25000	13.00000	4.33333
7	3.72727	5.12500	13.66667
Sums	43.49438	122.32379	144.44445

TAB. II. PART I.

N°	Ratios of the temperaments for equal harmony of the		
2	V + VIII	VI	5 : 1
8	V + VIII	3d	10 : 1
13	4th + VIII	VI	5 : 8
3	4th + VIII	3d	5 : 4
5	V + VIII	III	20 : 3
9	V + VIII	6th	40 : 3
10	4th + VIII	III	5 : 6
1	4th + VIII	6th	5 : 3
1	VI + VIII	III	2 : 3
6	VI + VIII	6th	4 : 3
12	3d + VIII	III	1 : 3
1	3d + VIII	6th	2 : 3
			3 dozen
8	V + VIII	VI + VIII	10 : 1
14	V + VIII	3d + VIII	20 : 1
3	4th + VIII	VI + VIII	5 : 4
1	4th + VIII	3d + VIII	4 : 2
4	V + VIII	III + VIII	10 : 3
15	V + VIII	6th + VIII	80 : 3
16	4th + VIII	III + VIII	5 : 12
4	4th + VIII	6th + VIII	10 : 3
12	VI + VIII	III + VIII	1 : 3
7	VI + VIII	6th + VIII	8 : 3
17	3d + VIII	III + VIII	1 : 6
6	3d + VIII	6th + VIII	4 : 3
	In two	Octaves	4 dozen

TAB. II. PART II.

N°	Reciprocals of the temperaments of the V, 4th & Comp.	VI, 3d & Comp.	III, 6th & Comp.
2	3. 20000	16. 00000	4. 00000
8	3. 10000	31. 00000	3. 44444
13	4. 60000	2. 87500	8. 33333
3	3. 80000	4. 75000	19. 00000
5	3. 85000	4. 52941	25. 66667
9	3. 92500	4. 24324	52. 33333
10	5. 20000	2. 36364	5. 66667
1	3. 40000	8. 50000	5. 66667
1	3. 40000	8. 50000	5. 66667
6	3. 57143	6. 25000	8. 33333
12	3. 25000	13. 00000	4. 33333
1	3. 40000	8. 50000	5. 66667
Sums	44. 69643	110. 51129	122. 11111
8	3. 10000	31. 00000	3. 44444
14	3. 05000	61. 00000	3. 21053
3	3. 80000	4. 75000	19. 00000
1	3. 40000	8. 50000	5. 66667
4	3. 70000	5. 28571	12. 33333
15	3. 96250	4. 11688	105. 66667
16	6. 40000	1. 88235	3. 33333
4	3. 70000	5. 28571	12. 33333
12	3. 25000	13. 00000	4. 33333
7	3. 72727	5. 12500	13. 66667
17	3. 14286	22. 00000	3. 66667
6	3. 57143	6. 25000	8. 33333
Sums	44. 80406	168. 19565	188. 98830

TAB. II. PART I.

N°	Ratios of the temperaments for equal harmony of the		
8	V	VI + 2VIII	10 : 1
14	V	3d + 2VIII	20 : 1
2	4th	VI + 2VIII	5 : 1
8	4th	3d + 2VIII	10 : 1
10	V	III + 2VIII	5 : 6
15	V	6th + 2VIII	80 : 3
16	4th	III + 2VIII	5 : 12
9	4th	6th + 2VIII	40 : 3
12	VI	III + 2VIII	1 : 3
18	VI	6th + 2VIII	32 : 3
17	3d	III + 2VIII	1 : 6
11	3d	6th + 2VIII	16 : 3
			5 dozen
14	V + VIII	VI + 2VIII	20 : 1
19	V + VIII	3d + 2VIII	40 : 1
1	4th + VIII	VI + 2VIII	5 : 2
2	4th + VIII	3d + 2VIII	5 : 1
1	V + VIII	III + 2VIII	5 : 3
20	V + VIII	6th + 2VIII	160 : 3
21	4th + VIII	III + 2VIII	5 : 24
5	4th + VIII	6th + 2VIII	20 : 3
17	VI + VIII	III + 2VIII	1 : 6
11	VI + VIII	6th + 2VIII	16 : 3
22	3d + VIII	III + 2VIII	1 : 12
7	3d + VIII	6th + 2VIII	8 : 3
			6 dozen

TAB. II. PART II.

N°	Reciprocals of the temperaments of the v, 4th & Comp.	vi, 3d & Comp.	iii, 6th & Comp.
8	3. 10000	31. 00000	3. 44444
14	3. 05000	61. 00000	3. 21053
2	3. 20000	16. 00000	4. 00000
8	3. 10000	31. 00000	3. 44444
10	5. 20000	2. 36364	5. 66667
15	3. 96250	4. 11688	105. 66667
16	6. 40000	1. 88235	3. 33333
9	3. 92500	4. 24324	52. 33333
12	3. 25000	13. 00000	4. 33333
18	3. 91429	4. 28125	45. 66667
17	3. 14286	22. 00000	3. 66667
11	3. 84211	4. 56250	24. 33333
Sums	46. 08676	195. 44986	243. 09941
14	3. 05000	61. 00000	3. 21053
19	3. 02500	121. 00000	3. 10256
1	3. 40000	8. 50000	5. 66667
2	3. 20000	16. 00000	4. 00000
1	3. 40000	8. 50000	5. 66667
20	3. 98125	4. 05732	212. 33333
21	8. 80000	1. 51724	2. 16667
5	3. 85000	4. 52941	25. 66667
17	3. 14286	22. 00000	3. 66667
11	3. 84211	4. 56250	24. 33333
22	3. 07692	40. 00000	3. 33333
7	3. 72727	5. 12500	13. 66667
Sums	46. 49541	296. 79147	302. 81310

TAB. II. PART I.

N°	Ratios of the temperaments for equal harmony of the		
1	V + 2VIII	VI	5 : 2
2	V + 2VIII	3^{d}	5 : 1
23	4^{th} + 2VIII	VI	5 : 16
13	4^{th} + 2VIII	3^{d}	5 : 8
4	V + 2VIII	III	10 : 3
5	V + 2VIII	6^{th}	20 : 3
16	4^{th} + 2VIII	III	5 : 12
10	4^{th} + 2VIII	6^{th}	5 : 6
12	VI + 2VIII	III	1 : 3
1	VI + 2VIII	6^{th}	2 : 3
17	3^{d} + 2VIII	III	1 : 6
12	3^{d} + 2VIII	6^{th}	1 : 3
			7 dozen
2	V + 2VIII	VI + VIII	5 : 1
8	V + 2VIII	3^{d} + VIII	10 : 1
13	4^{th} + 2VIII	VI + VIII	5 : 8
3	4^{th} + 2VIII	3^{d} + VIII	5 : 4
1	V + 2VIII	III + VIII	5 : 3
9	V + 2VIII	6^{th} + VIII	40 : 3
21	4^{th} + 2VIII	III + VIII	5 : 24
1	4^{th} + 2VIII	6^{th} + VIII	5 : 3
17	VI + 2VIII	III + VIII	1 : 6
6	VI + 2VIII	6^{th} + VIII	4 : 3
22	3^{d} + 2VIII	III + VIII	1 : 12
1	3^{d} + 2VIII	6^{th} + VIII	2 : 3
			8 dozen

TAB. II. PART II.

N°	Reciprocals of the temperaments of the		
	V, 4th & Comp.	VI, 3d & Comp.	III, 6th & Comp.
1	3. 40000	8. 50000	5. 66667
2	3. 20000	16. 00000	4. 00000
23	6. 20000	1. 93750	3. 18182
13	4. 60000	2. 87500	8. 33333
4	3. 70000	5. 28571	12. 33333
5	3. 85000	4. 52941	25. 66667
16	6. 40000	1. 88235	3. 33333
10	5. 20000	2. 36364	5. 66667
12	3. 25000	13. 00000	4. 33333
1	3. 40000	8. 50000	5. 66667
17	3. 14286	22. 00000	3. 66667
12	3. 25000	13. 00000	4. 33333
Sums	49. 59286	99. 87361	48. 18182
2	3. 20000	16. 00000	4. 00000
8	3. 10000	31. 00000	3. 44444
13	4. 60000	2. 87500	8. 33333
3	3. 80000	4. 75000	19. 00000
1	3. 40000	8. 50000	5. 66667
9	3. 92500	4. 24324	52. 33333
21	8. 80000	1. 51724	2. 16667
1	3. 40000	8. 50000	5. 66667
17	3. 14286	22. 00000	3. 66667
6	3. 57143	6. 25000	8. 33333
22	3. 07692	40. 00000	3. 33333
1	3. 40000	8. 50000	5. 66667
Sums	47. 41621	154. 13548	101. 61111

TAB. II. PART I.

N°	Ratios of the temperaments for equal harmony of the		
8	v + 2VIII	VI + 2VIII	10 : 1
14	v + 2VIII	3d + 2VIII	20 : 1
3	4th + 2VIII	VI + 2VIII	5 : 4
1	4th + 2VIII	3d + 2VIII	5 : 2
10	v + 2VIII	III + 2VIII	5 : 6
15	v + 2VIII	6th + 2VIII	80 : 3
24	4th + 2VIII	III + 2VIII	5 : 48
4	4th + 2VIII	6th + 2VIII	10 : 3
22	VI + 2VIII	III + 2VIII	1 : 12
7	VI + 2VIII	6th + 2VIII	8 : 3
25	3d + 2VIII	III + 2VIII	1 : 24
6	3d + 2VIII	6th + 2VIII	4 : 3
	In three	Octaves	9 dozen

TAB. III.

The numbers and sums of the -----		
Dozen	N8	v, 4th & Comp.
In 1 VIIIve. 1st	12	42. 72013
2d	12	43. 49438
3d	12	44. 69643
4th	12	44. 80406
In 2 VIIIves.	48	175. 71500
5th	12	46. 08676
6th	12	46. 49541
7th	12	49. 59286
8th	12	47. 41621
9th	12	53. 22812
In 3 VIIIves.	108	418. 53436

TABLE IV. Contains all the different temperaments for making every two concords, in the different parcels in 3 Octaves, equally harmonious.

N°	V 4th &c	VI 3d &c
1	$\frac{5}{17}c$	$\frac{2}{17}c$
2	5	$\frac{1}{16}$
3	5	$\frac{4}{19}$
4	10	$\frac{7}{37}$
5	20	$\frac{17}{77}$
6	7	$\frac{4}{25}$
7	11	$\frac{8}{41}$
	In one Octav	
8	10	$\frac{1}{31}$
9	40	$\frac{37}{157}$
10	5	$\frac{11}{26}$
11	19	$\frac{16}{73}$
12	4	$\frac{1}{13}$
13	5	$\frac{8}{23}$

I procals of all the different temperaments
63 Octaves.

8d & Comp.	III, 6th & Comp.	
8. 50000	3) 17	5. 66667
16. 00000	4) 16	4. 00000
4. 75000	1) 19	19. 00000
5. 28571	3) 37	12. 33333
4. 52941	3) 77	25. 66667
6. 25000	3) 25	8. 33333
5. 12500	3) 41	13. 66667
Octave		
31. 00000	9) 31	3. 44444
4. 24324	3) 157	52. 33333
2. 36364	— 6) 26	5. 66667
4. 56250	3) 73	24. 33333
13. 00000	3) 13	4. 33333
2. 87500	— 3) 23	8. 33333
61. 00000	19) 61	3. 21053
4. 11688	3) 317	105. 66667
1. 88235	— 12) 32	3. 33333
22. 00000	6) 22	3. 66667
Octaves		
4. 28125	3) 137	45. 66667
121. 00000	39) 121	3. 10256
4. 05732	3) 637	212. 33333
1. 51724	— 24) 44	2. 16667
40. 00000	12) 40	3. 33333
1. 93750	— 11) 31	3. 18182
1. 28302	— 48) 68	2. 58333
76. 00000	24) 76	3. 16667
Octaves		

TABLE IV. Contains all the different temperaments for making every two concords, in the different parcels in 3 Octaves, equally harmonious.

N°	V 4th &c	VI 3d &c	III 6th &c
1	5/17 c	2/17 c	3/17 c
2	5	1	4 c
		16	
3	5	4	1
		19	
4	10	7	3
		37	
5	20	17	3
		77	
6	7	4	3
		25	
7	11	8	3
		41	
	In one Octave		
8	10	1	9
		31	
9	40	37	3
		157	
10	5	11	—6
		26	
11	19	16	3
		73	
12	4	1	3
		13	
13	5	8	—3
		23	

N°	V 4th &c	VI 3d &c	III 6th &c
14	20	1	19
		61	
15	80	77	3
		317	
16	5	17	—12
		32	
17	7	1	6
		22	
	In two Octaves		
18	35	32	3
		137	
19	40	1	39
		121	
20	160	157	3
		637	
21	5	29	—24
		44	
22	13	1	12
		40	
23	5	16	—11
		31	
24	5	53	—48
		68	
25	25	1	24
		76	
	In three Octaves		

TABLE V. Contains the reciprocals of all the different temperaments in 3 Octaves.

N°	V, 4th & Comp.		VI, 3d & Comp.		III, 6th & Comp.	
1	5) 17	3.40000	2) 17	8.50000	3) 17	5.66667
2	5) 16	3.20000	1) 16	16.00000	4) 16	4.00000
3	5) 19	3.80000	4) 19	4.75000	1) 19	19.00000
4	10) 37	3.70000	7) 37	5.28571	3) 37	12.33333
5	20) 77	3.85000	17) 77	4.52941	3) 77	25.66667
6	7) 25	3.57143	4) 25	6.25000	3) 25	8.33333
7	11) 41	3.72727	8) 41	5.12500	3) 41	13.66667
		In	one	Octave		
8	10) 31	3.10000	1) 31	31.00000	9) 31	3.44444
9	40) 157	3.92500	37) 157	4.24324	3) 157	52.33333
10	5) 26	5.20000	11) 26	2.36364	— 6) 26	$\bar{5}$.66667
11	19) 73	3.84211	16) 73	4.56250	3) 73	24.33333
12	4) 13	3.25000	1) 13	13.00000	3) 13	4.33333
13	5) 23	4.60000	8) 23	2.87500	— 3) 23	$\bar{8}$.33333
14	20) 61	3.05000	1) 61	61.00000	19) 61	3.21053
15	80) 317	3.96250	77) 317	4.11688	3) 317	105.66667
16	5) 32	6.40000	17) 32	1.88235	—12) 32	$\bar{3}$.33333
17	7) 22	3.14286	1) 22	22.00000	6) 22	3.66667
		In	two	Octaves		
18	35) 137	3.91429	32) 137	4.28125	3) 137	45.66667
19	40) 121	3.02500	1) 121	121.00000	39) 121	3.10256
20	160) 637	3.98125	157) 637	4.05732	3) 637	212.33333
21	5) 44	8.80000	29) 44	1.51724	—24) 44	$\bar{2}$.16667
22	13) 40	3.07692	1) 40	40.00000	12) 40	3.33333
23	5) 31	6.20000	16) 31	1.93750	—11) 31	$\bar{3}$.18182
24	5) 68	13.60000	53) 68	1.28302	—48) 68	$\bar{2}$.58333
25	25) 76	3.04000	1) 76	76.00000	24) 76	3.16667
		In	three	Octaves		

TAB. II. PART II.

N°	Reciprocals of the temperaments of the v, 4th & Comp.	vi, 3d & Comp.	iii, 6th & Comp.
8	3. 10000	31. 00000	3. 44444
14	3. 05000	61. 00000	3. 21053
3	3. 80000	4. 75000	19. 00000
1	3. 40000	8. 50000	5. 66667
10	5. 20000	2. 36364	5. 66667
15	3. 96250	4. 11688	105. 66667
24	13. 60000	1. 28302	2. 58333
4	3. 70000	5. 28571	12. 33333
22	3. 07692	40. 00000	3. 33333
7	3. 72727	5. 12500	13. 66667
25	3. 04000	76. 00000	3. 16667
6	3. 57143	6. 25000	8. 33333
Sums	53. 22812	245. 67425	172. 07164

TAB. III.

------ reciprocals of the temperaments vi, 3d & Comp.	iii, 6th & Comp.
87. 47583	126. 33334
122. 32379	144. 44445
110. 51129	122. 11111
168. 19565	188. 98830
488. 50656	581. 87720
195. 44986	243. 09941
296. 79147	302. 81310
99. 87361	48. 18182
154. 13548	101. 61111
245. 67425	172. 07164
1480. 43123	1449. 65428

TAB. VI.

The values of Eq *and* Eq' *in Fig.* 45 and 46, ---
wards the temperament of the III[d], *for* ---
equally and the most harmonious.

In 1 Octave.

(*a*)
$$\frac{12}{42.72013} = 0.2808980 = GD = d$$
$$\frac{12}{87.47583} = 0.1371808 = AH = b$$
$$\frac{12}{126.33333} = 0.0949868 = EM = m$$

In 2 Octaves.

$$\frac{48}{175.71500} = 0.2731696 = GD = d$$
$$\frac{48}{488.50656} = 0.0982587 = AH = b$$
$$\frac{48}{581.87720} = 0.0824916 = EM = m$$

In 3 Octaves.

$$\frac{108}{418.53436} = 0.2580433 = GD = d$$
$$\frac{108}{1480.43123} = 0.0729517 = AH = b$$
$$\frac{108}{1449.65428} = 0.0745005 = EM = m$$

(*a*) See the last Table.

TAB. VI.

--- *being the first and second approximations to-*
--- *making all the concords in* 1, 2 *or* 3 *octaves*

Hence

$Ef = 4d - 1 = 0.1235920$

$Ei = \frac{1-4b}{3} = 0.1504256$

$EM = m = 0.0949868$

$3)\ 0.3690044$

$Eq = 0.1230015$

$Eq' = 0.122233.$ in 1 Octave,

See Tab. VII.

$Ef = 4d - 1 = 0.0926784$

$Ei = \frac{1-4b}{3} = 0.2023217$

$EM = m = 0.0824916$

$3)\ 0.3774917$

$Eq = 0.1258306$

$Eq' = 0.124719.$ in 2 Octaves.

$Ef = 4d - 1 = 0.0321732$

$Ei = \frac{1-4b}{3} = 0.2360634$

$Em = m = 0.0745005$

$3)\ 0.3427371$

$Eq = 0.1142457$

$Eq' = 0.11024..$ in 3 Octaves.

TAB. VII.

The computation of Eq′ *in Fig.* 46, *being the ---
ment of the* 11^{d}, *for making all the concords ---*

$$\frac{1}{GD} = \frac{1}{d} = \frac{42.72013}{12} = 3.560011$$

$$\frac{1}{Gg} = \frac{3}{1-b} = \frac{3}{0.8628192} = 3.476974$$

$$\frac{1}{Gk} = \frac{4}{1+m} = \frac{4}{1.0949868} = \underline{3.652913}$$

$$3\,)\ 10.689898$$

Arith. mean 3.563299

$$GD' = d' = \frac{1}{3.563299} = 0.280631$$

$$\frac{1}{AH} = \frac{1}{b} = \frac{87.47583}{12} = 7.289653$$

$$\frac{1}{Ae} = \frac{1}{1-3d} = \frac{1}{0.1573060} = 6.357031$$

$$\frac{1}{Al} = \frac{4}{1-3m} = \frac{4}{0.7150396} = \underline{5.494096}$$

$$3\,)\ 19.240780$$

Arith. mean 6.413593

$$AH' = b' = \frac{1}{6.413593} = 0.155919$$

TAB. VII.

--- *second approximation towards the tempera-*
--- *in* 1 *octave equally and the most harmonious.*

$$\frac{1}{EM} = \frac{1}{m} = \frac{126.33333}{12} = 10.527777$$

$$\frac{1}{Ef} = \frac{1}{4d-1} = \frac{1}{0.1235920} = 8.091138$$

$$\frac{1}{Ei} = \frac{3}{1-4b} = \frac{3}{0.4512768} = \underline{6.647805}$$

$$3\,)\ 25.266720$$

Arith. mean 8.422240

$$EM' = m' = \frac{1}{8.422240} = 0.118733$$

Hence $Ef' = 4d' - 1 = 0.122524$

$$Ei' = \frac{1-4b'}{3} = 0.125441$$

$$Em' = m' = \underline{0.118733}$$

$$3\,)\ 0.366698$$

$$Eq' = 0.122233$$

See the values of Eq' in 2 and 3 octaves in Tab. VI, part 2[d].

PRO-

PROPOSITION XVII.

A system of commensurable intervals deduced from dividing the octave into 50 equal parts, and taking the limma L = 5 of them, the tone T = 8 and consequently the lesser 3^d L + T = 13, the greater III^d 2 T = 16, the 4^th L + 2 T = 21, the V^th L + 3 T = 29, &c, according to the table of elements (a), will differ insensibly from the system of equal harmony: I mean with regard to the harmony of the respective consonances in both.

For since L = 5, T = 8 and the III^d 2 T = 16 and the VIII = 5 T + 2 L = 50, we have the III^d 2 T : VIII :: 8 : 25; whence the III^d 2 T = $\frac{8}{25}$ VIII = $\frac{8}{25}$ log. 2 = $\frac{8}{25}$ × 0. 30102. 99957 = 0. 09632. 95962, which subtracted from the perfect III^d = log. $\frac{4}{5}$ = 0. 09691. 00130, leaves the temperament 0.00058.04168, which is to the comma c = log. $\frac{81}{80}$ = 0. 00539. 50319 as 4 to 37 very nearly (b). Hence the tem-

(a) Prop. III.

(b) See an example of the like reduction in the next Scholium.

temperament of the III^d is $-\frac{4}{37}c$, and thoſe of the vth and VIth as in Tab. I, by prop. III. cor. I. 2. 3.

TABLE I.

T : L : : 8 : 5 VIII = 50	The ſyſtem of equal harmony.	Ratios of the temperaments and of the beats made in any given time (*c*).
V $-\frac{1}{4}c-\frac{1}{37}c$	V $-\frac{1}{4}c-\frac{1}{36}c$	$\frac{1}{4}+\frac{1}{37} : \frac{1}{4}+\frac{1}{36} :: 369 : 370$
VI $+\frac{1}{4}c-\frac{3}{37}c$	VI $+\frac{1}{4}c-\frac{3}{36}c$	$\frac{1}{4}-\frac{3}{37} : \frac{1}{4}-\frac{3}{36} :: 75 : 74$
III $-\frac{4}{37}c$	III $-\frac{4}{36}c$	$\frac{4}{37} : \frac{4}{36} :: 36 : 37$

Now though the concords of the ſame name in this ſyſtem and that of equal harmony are not exactly equally harmonious (*d*), and though the difference of an unit in the largeſt number of beats made in a given time may be diſtinguiſhed by counting them; yet if the numbers be not ſmaller than thoſe in the table, the difference in the harmony of the concords will be deemed inſenſible by proper judges; which are thoſe only that have carefully attended to the beats of concords in tuning inſtruments. But any one elſe may be ſatisfied experimentally, by cauſing

(*c*) Prop. XI. coroll. 4.
(*d*) Prop. XIII. coroll. 5.

causing two concords to the same base to beat as in the table. Q. E. D.

Scholium.

In like manner if $T = 5$ and $L = 3$, then the octave $5T + 2L$ is $= 31$ and the temperament of this system, which *Hugenius* has adopted (*e*), will be found as in the third column of the next table.

TABLE II.

T : L : : 2 : 1 VIII = 12	T : L : : 3 : 2 VIII = 19	T : L : : 5 : 3 VIII = 31
$\text{V} - \frac{1}{4}c + \frac{3}{19}c$	$\text{V} - \frac{1}{4}c - \frac{3}{35}c$	$\text{V} - \frac{1}{4}c + \frac{1}{110}c$
$\text{VI} + \frac{1}{4}c + \frac{9}{19}c$	$\text{VI} + \frac{1}{4}c - \frac{9}{35}c$	$\text{VI} + \frac{1}{4}c + \frac{3}{110}c$
$\text{III} \quad + \frac{12}{19}c$	$\text{III} \quad - \frac{12}{35}c$	$\text{III} \quad + \frac{4}{110}c$

On the contrary, if from the given temperament of a system it be required to find the ratio of T to L, we may proceed as follows. Let it be proposed to approximate to the system of equal harmony, where $2T = \text{III} - \frac{1}{9}c$ (*f*); then

(*e*) Cyclus harmonicus, at the end of his Works, or Histoire des Ouvrages des Sçavans, Octob. 1691, pag. 78.

(*f*) Prop. XVI. Scholium 2, Art. 9.

then ſince $5\text{T} + 2\,\text{L} = \text{VIII}$, we have $2\,\text{L} = (\text{VIII} - 5\text{T} =)\ \text{VIII} - \frac{5}{2} \times \overline{\text{III} - \frac{1}{9}c}$, whence $\text{T} : \text{L} :: \text{III} - \frac{1}{9}c : \text{VIII} - \frac{5}{2} \times \overline{\text{III} - \frac{1}{9}c}$.

To find this ratio, we have the $\text{III} = \log. \frac{5}{4} = 0.09691.00130$ and the comma $c = \log. \frac{81}{80} = 0.00539.50319$ and $\frac{1}{9}c = 0.00059.94480$. Whence $2\text{T} = \text{III} - \frac{1}{9}c = 0.09631.05650$ and $\frac{5}{2} \times \overline{\text{III} - \frac{1}{9}c} = 0.24077.64125$ and the $\text{VIII} = \log. 2 = 0.30102.99957$ and $2\,\text{L} = \text{VIII} - \frac{5}{2} \times \overline{\text{III} - \frac{1}{9}c} = 0.06025.35832$, and laſtly $\text{T} : \text{L} :: 963105650 : 602535832$.

Now the quotients of the greater term of this ratio divided by the leſſer and of the leſſer divided by the remainder and of the former remainder by the latter &c, are 1, 1, 1, 2, 24, &c. Whence the ratios greater than the true one are 2 to 1, 5 to 3, 8 to 5, &c, and the leſſer are 3 to 2, 11 to 7, &c (*g*).

Hence taking T to L ſucceſſively in thoſe ratios, by the method uſed in the demonſtration of the propoſition, the temperaments of the approximating rational ſyſtems will be found as in the tables. By which we ſee how much and which

(*g*) See Mr. *Cotes*'s Harmonia Menſurarum, Schol. 3. prop. 1.

which way they differ from that of mean tones, as well as from that of equal harmony in Table 1.

SECTION VIII.

The ſcale of muſical ſounds is fully explained and made changeable upon the harpſichord, in order to play all the flat and ſharp ſounds, that are uſed in any piece of muſic, upon no other keys than thoſe in common uſe.

DEFINITIONS.

I. The interval of a perfect octave being divided, in any tempered ſyſtem, into 5 equal tones and 2 equal limmas (*b*), the exceſs of the tone above the limma is called a Minor limma.

II. The difference of the major and minor limma is called a Dieſis.

III. If the difference of the intervals of two conſonances to the ſame baſe be a dieſis, I ſhall call either of them a Falſe conſonance when ever, in playing on the organ or harpſichord, it is ſubſtituted for the other which ought to be uſed; as it often is for want of a complete ſcale of ſounds in thoſe inſtruments.

IV. The notes *A*※, *B*※, &c, ſignify ſounds which are ſharper, and *A*♭, *B*♭, &c, ſounds which

(*b*) See ſect. IV art. 3, or the dem. of prop. 2, or prop. 3.

which are flatter by a minor limma than the respective primary sounds A, B, &c: And A^{**}, A^{bb}, &c, signify sounds whose distance from A is double the distance of A^{*} or A^{b} from A and alike situated.

1. Pl. XVIII. Fig. 48 or 49. The interval of a perfect octave being represented by the circumference of any circle (*i*) and supposed to be divided by the sounds A, B, C, D, E, F, G into 5 tones and 2 limmas, towards the acuter sounds take the interval AA^{*} equal to the minor limma $AB—BC$, and towards the graver take AA^{b} equal to AA^{*}, and when the like flat and sharp sounds are placed at that distance on each side of the other primary sounds B, C, D, E, F, G, every tone will be divided by a flat or a sharp sound into a major and a minor limma, and by both into two minor limmas with a diesis between them; and each primary limma, BC, EF, will be divided by a flat or a sharp sound into a minor limma and a diesis, and by both into two dieses with an interval between them.

2. Fig. 48. In the *Hugenian* system the octave is divided into 31 equal parts, of which the tone is 5, the major limma 3, the minor 2 and the diesis 1 (*k*).

Fig. 49. In the system of Equal Harmony the octave is divided into 50 equal parts, of which the tone is 8, the major limma 5, the minor 3 and the diesis 2 (*l*).

Therefore the former tone is to the latter as

L $\frac{5}{31}$

(*i*) Sect. IV. art. 7. (*k*) Prop. XVII, schol.
(*l*) Prop. XVII.

$\frac{5}{31}$ VIII to $\frac{8}{50}$ VIII, or as 125 to 124, and the former diesis is to the latter as $\frac{1}{31}$ to $\frac{2}{50}$, or 25 to 31; and since a quarter of a comma is about $\frac{1}{223}$ VIII (*m*) the former diesis $\frac{1}{31}$ VIII contains above $\frac{7}{4}$, and the latter almost $\frac{9}{4}$ of a comma.

3. Fig. 48 or 49. In either of those systems or any other of that kind, by going many times round the circle it will appear, that in ascending from *F* continually by v^ths the 7 primary notes will first occur in this order *FCGDAEB*, and then recur once sharpened in the same order, and again twice shapened &c: Likewise in descending from *F* by v^ths, they will recur once flattened in that order thus inverted, *Bb Eb Ab Db Gb Cb Fb*, and again twice flattened &c: And these several cycles joined together make the following progression ascending by v^ths; *Ebb Bbb*, *Fb Cb Gb Db Ab Eb Bb*, *F C G D A E B*, *F※ C※ G※ D※ A※ E※ B※*, *F※※ C※※* &c.

4. Hence a Table of the minor and major consonances to any number of Keys or base notes in that progression placed in the first column (*n*), is thus deduced. Opposite to any Key as *D* write the 12 trebles *Eb*, *E*, *F*, *F※*, &c of the minor and major consonances within the VIII in the order of their marks, 2^d, II^d, 3^d, III^d, &c at the top of the table, which trebles are found by going round the circle; then place the same progression of v^ths above

(*m*) Found by dividing the log. of 2 by $\frac{1}{4}$ log. $\frac{81}{80}$.

(*n*) Plate XIX.

above and below the treble E^b in col. 2, as ſtands above and below E^b in col. 1; and having done the like to the other trebles E, F, &c, the table is finiſhed.

For ſince the interval $D\,A$ in col. 1 is equal to $E^b\,B^b$ in col. 2, it follows that in col. 1 and 2 the interval $A\,B^b$ equals $D\,E^b$; and the ſame may be ſaid of the reſt of the table. At the bottom of it the letters L, l, D ſignify the major and minor limma and the dieſis, as being the differences of the intervals marked at the top.

5. As the organ or harpſicord has but 12 ſounds in the octave, whoſe notes are F, C, G, D, A, E, B, with F※, C※, G※, above, and E^b, B^b, below them in col. 1; all the notes below E^b in col. 1 and in thoſe of the minor conſonances, and all above G※ in col. 1 and in thoſe of the major conſonances have no ſounds anſwering to them in thoſe inſtruments; and are therefore excluded, or diſtinguiſhed from the notes that have ſounds, by circles round them, both in the table and in Fig. 48 and 49.

Conſequently when any of the excluded notes D※, A※, E※, B※, F※※, C※※, that are above G※, occur in a piece of muſic, as moſt of them often do, the muſician is obliged to ſubſtitute for them the ſounds of E^b, B^b, F, C, G, D, reſpectively, which being higher by a dieſis (*o*) make falſe conſonances (*p*).

(*o*) As appears by Fig. 48, or by the collateral notes in the columns of 4ths and 5ths in the table of conſonances.

(*p*) Def. III.

Likewiſe when any of the excluded notes *Ab*, *Db*, *Gb*, *Cb*, *Fb*, *Bbb*, that are below *Eb*, occur, as ſome of them often do, the muſician muſt ſubſtitute for them *G*※, *C*※, *F*※, *B*, *E*, *A*, reſpectively, which being lower by a dieſis make falſe conſonances.

Hence the two middlemoſt Keys *D*, *A* have one falſe conſonance in each, and the numbers of them in the ſucceſſive higher or lower Keys, increaſe in the arithmetical progreſſion 2, 3, 4, 5, 6. Whence it is eaſy to collect that ſeven twenty-fourths of the whole number of major and minor conſonances in the ſcale of the organ or harpſichord, are falſe; beſides a larger proportion of falſe ones among the Superfluous and Diminiſhed conſonances hereafter mentioned.

6. The conſonances to all the Keys above *E* have no flat notes; becauſe *Bb* is the higheſt flat note in every column of minor conſonances, and is the higheſt of all where it is the minor 5th to the Key *E*. Again, the conſonances to all the keys below *C* have no ſharp notes; becauſe *F*※ is the loweſt ſharp note in every column of major conſonances, and is the loweſt of all where it is the major IVth to the Key *C*. Therefore the conſonances to thoſe two and the intermediate Keys, *CGDAE*, have both flat and ſharp notes among them.

Hence it comes to paſs that the concords (*q*) to the 4 middlemoſt keys *G*, *D*, *A*, *E*, which are the

(*q*) Sect. III. art. 11th.

the open ſtrings of the violin, are all true, but not all the diſcords.

7. By adding a ſound for *A*♭, every one of the 6 lower keys *E*♭, *B*♭, *F*, *C*, *G*, *D*, will have one falſe conſonance changed into a true one, as appears by inſpection of the oblique diagonal rows of *A*♭ in the table. Likewiſe by adding another ſound for *D*※ every one of the 6 higher keys *A*, *E*, *B*, *F*※, *C*※, *G*※, will have one falſe conſonance changed into a true one. Now in this inlarged ſcale of 14 keys all the conſonances to *D* and *A*, the two middlemoſt, are true. And a like advantage will follow from giving ſounds to *D*♭ and *A*※, the two next exterior keys, and ſo forth.

Therefore univerſally, the number of the middlemoſt keys to which all the minor and major conſonances are true, is equal to the whole number of keys or ſounds in the octave diminiſhed by 12; ſo that the 24 ſounds in col. 1. would be neceſſary to make all theſe conſonances true in the 12 middlemoſt keys.

8. But beſides the major and minor conſonances in the Table there are others in the ſcale of Fig. 48 or 49, which I think are called Superfluous and Diminiſhed conſonances.

The interval of a major conſonance augmented by a minor limma makes the interval of a ſuperfluous conſonance; and the interval of a minor conſonance diminiſhed by a minor limma makes the interval of a diminiſhed conſonance.

Thus the treble of a ſuperfluous II^{d}, III^{d}, IVth, Vth, VIth, VIIth, &c to the key *F*, is *G*✳, *A*✳, *B*✳, *C*✳, *D*✳, *E*✳, reſpectively; and the treble of the diminiſhed 2^{d}, 3^{d}, 4th, 5th, 6th, 7th, &c to the key *B*, is *C b*, *D b*, *E b*, *F b*, *G b*, *A b*. And the like is to be underſtood in the other keys, where the trebles are often double ſharp and double flat ſounds; but are all omitted in the Table to avoid confuſion by adding ſo many notes to it.

9. I have heard of but one method of ſupplying the organ or harpſichord with more ſounds in each octave; which is by adding pipes or ſtrings for *A b*, *D b*, &c, and dividing the keys of their ſubſtitutes *G*✳, *C*✳, &c, each into two keys, the longer of them for ſounding *G*✳, *C*✳, &c as uſual, and the ſhorter for ſounding *A b*, *D b*, &c: and by doing the like for *D*✳, *A*✳, &c. But this method of ſupplying the defects of the ſcale is quite laid aſide, on account of the great difficulty in playing upon ſo many keys without extraordinary practice, and the following palliative remedy is univerſally received.

Pl. XVIII. Fig. 48 or 49. The octave being always divided into 5 tones and two limmas; by increaſing the tones equally till each becomes double the diminiſhing limma *BC* or *EF*, the dieſis, or difference between the major and minor limma, will be contracted to nothing, which by Defin. III annihilates all the falſe conſonances. But the harmony in this ſyſtem of 12 Hemitones is extremely coarſe and diſagreeable.

For

For the temperaments of the Vth and 4th, VIth and 3^{d}, III^{d} and 6th and their compounds with VIII^{ths}, are nearly $\frac{1}{10}$, $\frac{7}{10}$ and $\frac{6}{10}$ of a comma reſpectively (*r*) and in the ſyſtem of equal harmony they are $\frac{5}{18}$, $\frac{1}{6}$ and $\frac{1}{9}$ (*s*); by which ſyſtem, as being the moſt harmonious, all other ſyſtems ought to be examined, as by a ſtandard. Now $\frac{1}{10}$ being much leſs than $\frac{5}{18}$, makes the concords in the firſt parcel (*t*) finer than they ought to be; and $\frac{7}{10}$ and $\frac{6}{10}$ being much greater than $\frac{1}{6}$ and $\frac{1}{9}$, make the concords in the other two parcels much coarſer than they ought to be, the two leaſt of thoſe temperaments being as great as thoſe concords can properly bear.

Now for want of another ſound to terminate each dieſis in the ſcale, it is neceſſary in the tuning to diminiſh the dieſis till one ſound may ſerve tolerably for the other, and thus to approach towards that inharmonious ſyſtem of 12 hemitones, till the harmony of the ſcale becomes very coarſe before the falſe conſonances are barely tolerable (*u*).

L 4 9. That

(*r*) Prop. XVII. Tab. II^{d}. col. 1.
(*s*) Prop. XVI. ſchol. 2. art. 10 and 13.
(*t*) Prop. III. ſchol.
(*u*) This is done by ſharpening the major III^{ds} more than the ear can well bear, which inlarges the tones and leſſens the major limmas and dieſes: or, becauſe any 6 tones or 3 major III^{ds} and a dieſis (as $A\flat C + CE + EG{\divideontimes} + G{\divideontimes} A\flat$) make up the octave or circumference in Fig. 48.

9. That this is a bad expedient for ſupplying the want of more ſounds, is farther evident from the *Hugenian* ſyſtem, where the temperament common to the VIth and 3^{d}, &c being $\frac{1}{4}+\frac{3}{110}$ of a comma (x) is conſiderably greater than it ought to be, that is, than $\frac{1}{6}$ of a comma, as in the ſyſtem of equal harmony; and yet the *Hugenian* dieſis is $\frac{7}{4}$ of a comma (y), which being conſidered as a temperament of the falſe conſonances and being ſo much greater than $\frac{1}{4}+\frac{3}{110}$ of a comma muſt needs make horrible diſſonance.

10. Having therefore been long diſſatisfied with the coarſeneſs of the harmony even of the true conſonances in the ſcale of our preſent inſtruments, which is ſo defective too that not above a ſeventh or eighth part of the beſt compoſitions made ſince *Corelli*'s time, nor above a third or fourth of his can be played upon it without uſing many falſe conſonances; and being ſtill more diſguſted when theſe come into play, as they often do in the remaining two thirds or three fourths of *Corelli*'s works, and ſix ſevenths or ſeven eighths of all the reſt; I was glad to find out a better remedy for both thoſe defects; at leaſt in a ſcale of ſingle ſounds.

11. The ſtrings of the fore uniſon of the harpſichord being tuned as uſual to the notes of the common ſcale in the following lower line, let the ſounds

(x) Prop. XVII. Tab. 2. col. 3.
(y) Sect. VIII. art. 2.

ſounds of the back uniſon be altered to the notes in the upper line, each of which differs from the note under it by a dieſis (*z*).

A^b B^{bb} *A*※ C^b *B*※ D^b *C*※※ *D*※ F^b *E*※ G^b *F*※※

G※ *A* B^b *B* *C* *C*※ *D* E^b *E* *F* *F*※ *G*

Now ſince the jacks which ſtrike the ſtrings of any of theſe couples of notes, as *G*※ and A^b, ſtand both upon one key, by moving a ſtop hereafter deſcribed, that key can ſtrike either ſtring alone without ſounding the other: And ſince both the ſounds in any couple are ſeldom or never uſed in any ſingle piece of muſic, the muſician before he begins to play it, can put in, by the ſtop, that ſound which he ſees moſt occaſion for; and either of them being ſtruck by the ſame key, the execution is always the ſame as uſual.

For example, if beſides the ſounds *F*※, *C*※, *G*※ in the common ſcale, *D*※, *A*※, *E*※, *B*※, *F*※※ ſhould alſo occur in a piece of muſic (*a*) move their ſtops, and their ſtrings will be ſtruck by the keys of E^b B^b *F* *C* *G* reſpectively, whoſe ſounds are uſually ſubſtituted for the ſounds required.

12. A muſician by caſting his eye over any piece of muſic, can ſoon ſee what flat or ſharp ſounds are uſed in it which are not in the common ſcale; and to ſave that trouble for the future, may write them down at the beginning of the piece. Now and then it may be proper to obſerve whether

(*z*) As appears by Fig. 49.
(*a*) As in *Corelli's* xith ſolo and *Carbonel's* IIId &c.

ther the outermoſt of them in their progreſſion by v^{ths}, ſhould be put in or not, leſt its ſubſtitute ſhould occur oftener than the principal ſound itſelf. If both occur, that which recurs oftener muſt be in the ſcale. But as both occur very ſeldom the matter is ſcarce worth notice.

13. To ſhew by inſpection which are the falſe conſonances in the Harpſichord after any flat or ſharp ſounds are put into it by the ſtops; imagine the two middlemoſt tranſverſe parallelograms in the Table (*b*) and alſo the circles ſurrounding the notes which are not in the common harpſichord, to be drawn with the point of a diamond upon a pane of glaſs laid over them. Then if the ſounds of *D*※ and *A*※ for inſtance be put into the harpſichord, move the pane two lines higher till the uppermoſt line of the two parallelograms juſt takes in thoſe two notes in col. 1, and in this poſition the circles upon the pane will cover all thoſe notes in the table which are not in the preſent ſcale of the harpſichord, and point out the falſe conſonances to every key.

14. Thus you ſee how to make any given key as *E* or *B*, as free from falſe conſonances as *D* or *A* is in the common fixed ſcale; namely by putting in by the ſtops as many ſharp notes above *G*※ in the column of keys as ſhall bring *E* and *B* into the middle of the 12 keys then in inſtrument. And the like may be done for any given key below *D* by putting in flat notes below *E*♭.

And

(*b*) Plate XIX.

And thus a muſician that tranſpoſes muſic at ſight can accompany a voice with the pureſt and fineſt harmony in the propereſt key for the pitch of the voice. I ſay the fineſt harmony; becauſe this changeable ſcale may eaſily be tuned to the moſt harmonious ſyſtem (*c*) which is impracticable upon the common fixed ſcale, becauſe the dieſis would be ſo large as to render the falſe conſonances inſufferably bad (*d*).

15. The famous *Ruckers* and other muſicians of a delicate ear, always valued the tone of a ſingle ſtring for its diſtinctneſs and clearneſs, ſpirit and duration, and preferred it to that of uniſons and octaves. I muſt confeſs I have long been of that opinion, even before I thought of this changeable ſcale of ſingle ſounds, which however after ſome years experience upon my own harpſichord has fully confirmed me in it.

16. Uniſons by themſelves or with an octave are indeed an addition to the loudneſs of the tone, but not nearly in proportion to the number of ſtrings. Firſt becauſe the oppreſſion of the belly of the inſtrument by the force of ſo many ſtrings, hinders the facility and duration of its tremors; and ſecondly becauſe in tuning uniſons or octaves, it is manifeſt that their tone is never clear, loud and flowing, like that of a ſingle ſtring, except when they are preciſely perfect. But as this perfection continues but a very little time, eſpecially after

(*c*) Prop. XVI. ſchol. 2. art. 10 and 13.
(*d*) Sect. VIII. art. 2.

after the room is warmed by company, that clear ſinging tone is ſoon deſtroyed.

The compound tone of uniſons by themſelves, or with an octave, being of itſelf ſo indiſtinct, what beating and jarring muſt reſult from their complicated mixtures in playing three or four parts of muſic? Eſpecially as the imperfections of the uniſons and octaves in the courſe of playing are frequently added to the temperaments of the other conſonances, which if they were perfect could not bear thoſe imperfections ſo well as the uniſons and octaves do when ſounded by themſelves (*e*). This confuſed noiſe, like that of a dulcimer, is but too plainly perceived when the ear is held over the ſtrings of the harpſichord; and ſince it reſults from the multiplicity of ſtrings, it appears that the beſt way to improve this inſtrument is to find out methods for increaſing the ſtrength and clearneſs of the tone of ſingle ſtrings.

17. To me, who ſeldom hear any other than the ſingle ſtrings of my own harpſichord, the tone is as loud as I deſire, not only for leſſons and cantatas but alſo concertos accompanied with inſtruments in a large room. This indeed is more than a perſon could expect who has ſeldom or never attended to the tone of ſingle ſtrings except in the ſhort *pianos* after the long continued *fortes* upon the full harpſichord. The reaſon is that the very ſame objects affect our ſenſes very differently in different circumſtances, as is very evident in attending to any other ſenſation as well as that of ſounds.

(*e*) Prop. XIII. coroll. 8.

ſounds. "For inſtance, in coming out of a ſtrong "light into a room with the window-ſhutters al- "moſt cloſed, we immediately have a ſenſation "of darkneſs or a very little light, and this con- "tinues much longer than the pupil requires to "dilate and accommodate itſelf to that weak de- "gree of light, which is almoſt inſtantaneouſly "done. But after ſtaying ſome time in the ſame "or a much darker place, the ſame room which "appeared dark before, will be ſufficiently light." This obſervation is plainly applicable to ſounds, and more of them upon the other ſenſes may be ſeen in Dr. *Jurin*'s Eſſay on diſtinct and indiſtinct Viſion at the end of my Optics (*f*).

18. *An expedient for changing the ſounds of any harpſichord ready made, whereby to experience the truth of the foregoing obſervations.*

Pl. XXVI. The 66th figure repreſents the heads *t*, *u* of two jacks ſtanding as uſual upon one key, with their pens pointing oppoſite ways under the ſtrings on each ſide of them, as *G** and *A*♭, the back uniſon being raiſed to *A*♭. And *abcd* repreſents a ſmall braſs ſquare of the ſize in the figure, whoſe ſhorter leg *ab* is made very thin and placed between the jacks with its flat ſides facing them; and the longer leg *bced*, being placed directly over, and parallel to the next couple of ſtrings that are cloſeſt together, is filed four

(*f*) Art. 267.

four ſquare, and ſlides lengthways in two ſquare notches at *c* and *d* made in the parallel ſides *fcg*, *hdi* of a long braſs plate turned up like the ſides of a long ſhallow trough, which is ſupported a little above the ſtrings by a row of ſmall braſs pillars placed between the larger intervals of the ſtrings, as at *r*, *s*, &c, (but farther aſunder) and ſkrewed faſt into the pinboard of the harpſichord.

Theſe pillars have long necks paſſing through the holes *r*, *s*, &c in the bottom of the trough, and the nuts *r*, *s*, &c are ſkrewed upon the necks down to the bottom, to hold it faſt upon the ſhoulders of the pillars. And a braſs lid *FGHI* with oblong holes *R S*, &c correſponding to *r*, *s*, &c, being laid upon the trough *fg hi*, the upper nuts *R*, *S*, &c muſt be ſkrewed upon the ſame necks, to keep the lid tightiſh upon the longer leg of the ſquare *abcd* and others of the ſame ſize. A ſlit *mn* is made in the lid for a ſhort round pin *e* in the longer leg *cd* to come thro' it, and to move in it to and fro by a touch of the finger laid upon the pin. There muſt be as many ſuch ſquares as keys or couples of jacks, and the trough and lid may be each of one piece or conſiſt of two or three pieces joined together at the necks of the pillars or any where elſe.

While the jacks *t*, *u* are kept at their full height by holding down their key, with your finger laid upon the pin *e* puſh the leg *ab* againſt the far jack and mark the edge, or inner ſide of it with a line drawn cloſe by the upper edge of the

the leg *ab*; and after the ſquare is drawn back, make ſuch another mark upon the edge of the near jack. Then from a ſmall ſlender pin cut off a piece of a proper length meaſured from the point, and taking hold of its thicker end with a pair of pliers, preſs the point into the inner edge of the jack, a little above the mark and far enough to ſtick faſt in it, and do the like to the oppoſite jack. Let each pin project from its jack about a quarter of the ſpace between the two jacks, leaving about half of it void in the middle between the oppoſite ends of the pins, as repreſented in the figure.

Now when the two jacks are again raiſed by their key and kept at their full height, by drawing the ſquare backwards with your finger laid upon the pin *e* in the longer leg, the ſhorter leg *ab* will come under the pin in the near jack, and keep it ſuſpended with its pen above the ſtring G*, which therefore will be ſilent while the far jack plays alone upon the ſtring *A*♭; or, by puſhing the ſquare forward with your finger at *e*, the leg *ab* will go under the pin in the far jack, and ſuſpend its pen above the ſtring *A*♭, while the near jack plays alone upon the ſtring G *.

When all the ſtrings of the back uniſon are tuned to the notes in the upper line in art 11th all their jacks muſt be ſuſpended on the ſhorter legs of the ſquares; and then all the fore jacks will ſtrike the ſounds of the vulgar ſcale; and when other flat or ſharp ſounds are required in any piece of muſic, they muſt firſt be introduced by hold-

ing

ing down the keys of their uſual ſubſtitutes, one by one, and by drawing back the correſponding ſquares with a finger laid upon their pins at *e*. So long as you chooſe to play upon this changeable ſcale, keep the knobs of the right-hand ſtops of a double harpſichord tyed together by a ſtring.

When the ſtrings are tuned uniſons again, you may play upon them without removing this mechaniſm, provided you firſt draw every pin *e* towards the middle of the ſlit *mn*, in the lid *FI*, till it be oppoſite to the angular notch *o*, and then draw the lid lengthways by the button *p*, till the notch *o* embraces the pin *e* and keeps the ſhorter leg *ab* in the middle of the void ſpace between the ends of the pins in the oppoſite jacks: otherwiſe theſe pins may ſometimes ſtrike againſt the ſhorter legs of the ſquares. If that middle ſpace be too narrow, try whether it may not be widened a little by ſeparating the ſliders with ſome very thin wedges put between them: perhaps a little may be planed off from the back edges of the ſliders without hurting them.

I have deſcribed this mechaniſm ſo fully, I think, that any man who works true in braſs may eaſily apply it at a ſmall expence to any harpſichord ready made, and take it quite away without the leaſt damage to the inſtrument. I have uſed it ſome years in my own harpſichord with great pleaſure and no other inconvenience than that of removing the muſic book in order to touch the pins in the braſs ſquares behind it. But the following mechaniſm for the reception of which a little preparation muſt be made in the fabric of a new harpſichord, is quite free from that inconvience, and changes any ſound together with all its octaves in an inſtant, without putting down their keys.

19. *To make a new harpſichord wherein the ſounds being changeable at pleaſure, the uſual ſet of keys ſhall immediately ſtrike the proper ſcale for any propoſed piece of muſic.*

Pl. XXVI. Fig. 67. Conceiving the pins, ſtrings and jacks which in every octave belong to the notes *A*, *B*, *D*, *E*, to be taken away from the fore uniſons of a common harpſichord, the remaining pins, ſtrings and jacks will be ſufficient for the new harpſichord. Let the ſounds of theſe ſtrings be altered to the notes here placed by the ſides of their pins, and let theſe notes be written on the pin board of the new harpſichord; and that the tones of the ſtrings ſounded by the jacks in each row, may be as like each other as poſſible, let the tongues of the new jacks be put as near as may be to their inner edges, and theſe oppoſite edges be placed in the new ſlider as near as may be to one another, as repreſented in the figure.

Each of the keys, *A*, *B*, *D*, *E*, that moves but one jack (which therefore muſt be made as heavy with lead as two of the other jacks) ſtrikes always one and the ſame ſtring. But each of the 8 remaining keys, *Bb*, *C*, *C**, *Eb*, *F*, *F**, *G*, *G**, which moves a couple of jacks, is intended to ſtrike either of their ſtrings alone at pleaſure; that is, *A** or *Bb*, *B** or *C*, *C** or *Db*, *D** or *Eb*, *E** or *F*, *F** or *Gb*, *F*** or *G*, *G** or *Ab*.

Pl. XXVIII, Fig. 70. I think the beſt way to do this would be to have eight ſtops or braſs knobs ſkrewed as uſual on the ſhanks of eight draught irons made moveable in eight ſlits cut in the fore board of the new harpſichord: But to ſave a quarter of the labour and expenſe I propoſe to do it almoſt as well with only ſix, that is, three at each end of the fore board, as in the figure: where the notes of each couple of the changeable ſounds are written on oppoſite ſides of each knob, to the intent that the ſound or ſtring ſignified by this or that note *to which the knob is puſhed*, may be ſtruck alone by the key belonging to both the notes, while the other ſtring is ſilent. And ſince eight ſounds are intended to be changed by ſix knobs, each extreme knob is deſigned to change two ſounds at one puſh of it towards either couple of notes at the end of the ſlit, according to the ſame rule as before.

Hence by puſhing the two outermoſt knobs at the baſe end of the fore board, towards the right hand, and all the reſt towards the left, the keys will ſtrike the eight changeable ſounds in the vulgar ſcale, namely *F**, *C**, *G**, *Eb*, *Bb*, *F*, *C*, *G*, to be occaſionally changed by puſhing the knobs the contrary way: the other four, *A*, *B*, *D*, *E*, are fixt ſounds.

The

The notes of the changeable ſounds are placed in ſuch an order, that the ſounds belonging to the notes on the ſame ſides of the ſucceſſive knobs, continually aſcend or deſcend by v^{ths}, as in the Table of keys and conſonances in Plate XIX. Becauſe this order will be found much more convenient for altering and adapting the Scale to different pieces of muſic, than the alphabetical order of the ſame notes.

Now this deſign may be executed as follows.

Pl. XXVI. Fig. 67. When the pens for the back uniſons are put under their ſtrings, as denoted in the figure, and thoſe for the fore uniſons are drawn off from theirs by the ſtops of the common harpſichord, the jack holes in the two parallel rows have the ſame ſituation with reſpect to each other as they are intended to have in the new ſlider, except as already obſerved that the ſpace between the two rows ſhould be much narrower than in the common harpſichord.

By theſe directions if an accurate draught of all the jack holes be made upon a long braſs plate, part of which draught is repreſented in the figure, it may ſerve as a general pattern for making the new ſliders, or at leaſt to give a clear conception of their dimenſions, which a workman may execute in what manner he pleaſes. In order thereto let ſix braſs plates well flatted by a mill be made equal to each other in all their dimenſions. Let the length of each be equal to, or rather longer at firſt than that of a common ſlider, and the breadth of each be ſufficient for leaving a pretty ſtrong margin on the outſides of the jack holes, and then the thickneſs, after the work is finiſhed and poliſhed on both ſides, need not exceed a twelfth of an inch.

Fig. 67. In the 1ſt plate or ſlider the oppoſite holes for the jacks *A*※ and *B♭* and all their octaves, being made equal to thoſe in the general pattern, let all the reſt be made wider on each ſide, than thoſe in the pattern, by a twelfth of an inch, as repreſented at N° 1, below fig. 67.

In the 2d ſlider the holes for the jacks *B*※ and *C* and, to ſave another ſlider, for *F*※※ and *G* being made equal to thoſe in the pattern, let all the reſt be made wider on each ſide, than thoſe in the pattern, by $\frac{1}{12}$ of an inch, as at N° 2.

In the 3d ſlider the holes for the jacks *C*※ and *D♭* and, to ſave another ſlider, for *F*※ and *G♭* being made equal to thoſe in the pattern, make all the reſt wider on each ſide by $\frac{1}{12}$ of an inch.

In the 4th ſlider the holes for the jacks *D*※ and *E♭*, and in the 5th ſlider for *E*※ and *F*, and in the 6th ſlider for *G*※ and *A♭* being made equal to thoſe in the pattern, let all the reſt in each ſlider be made wider on each ſide, than thoſe in the pattern, by $\frac{1}{12}$ of an inch.

Pl. XXVII. Fig. 68. An under ſocket *pq* being made of wood as uſual, that is, with jack holes directly oppoſite to one another but nearer together as already obſerved, and being placed as near to the keys as may be, let an upper ſocket *rs* be made

made of braſs, exactly equal to the pattern in fig. 67 but without the notches for the tongues to play in; in which ſocket let every jack hole except for *A*, *B*, *D*, *E*, be made wider on its left ſide only by $\frac{1}{12}$ of an inch.

Leave $\frac{6}{12}$ or half an inch in height above this upper ſocket *r s*, for the ſix ſliders numbered 1, 2, 3, 4, 5, 6 as above, to lie upon it, and let all be ſupported as uſual by croſs pieces of boards fixed underneath.

All the jacks being put thro' their holes, this 68th figure repreſents a view of their edges and of the widening of the holes on each ſide, as they would appear to a diſtant eye placed at the fore end of the harpſichord, ſuppoſing the fore board and fore margin of the ſockets and ſliders were taken away.

When the two ſockets *pq* and *rs* are ſo adjuſted in their places that the jacks *A*, *B*, *D*, *E* ſtand upright and ſtrike their ſtrings, fix the ſockets in that poſition at one end only, that the ſhrinking or ſwelling of the harpſichord may not bend or ſtrain them. Then puſh all the ſliders towards the right hand, till the pens on the right hand of the other jacks in the far row ſhall ſtrike their ſtrings too. ſee alſo fig. 67.

Then if any ſlider be drawn back again, which the widened holes will permit, it will draw back the jacks in its narrow holes only, without ſtirring the reſt, and bring the right hand pen of the far jack from under its ſtring on the right hand and put the left hand pen of the near jack under its ſtring on the left hand; and then this latter ſtring will be ſounded alone by the ſame key while the former is ſilent.

The holes in the ſliders for the jacks *A*, *B*, *D*, *E*, which have no motion ſideways, need be widened on their right ſides only, as repreſented by the ſhades, to make room for the ſliders to move towards the left hand; but if they be widened on both ſides, according to the general direction above, no inconvenience will follow from it.

According to the uſual breadth of harpſichords the compaſs of our ſcale may conveniently be from double *G* up to *e* in *alt*.

Pl. XXVIII. Fig. 69. When the ſix ſliders are laid upon one another in any order, provided they coincide in length and breadth (and keep ſo by two pins put thro' two columns of holes at their ends) three round holes muſt be drilled through them all in the vacant places at *k* and *l*, oppoſite to the jacks *b* and *d* in *alt*, and at *m* a little above *e* in *alt*; and the hole at *k* in the 1[ſt] ſlider, at *l* in the 5[th], as numbered above, and at *m* in the 2[d] remaining round, all the reſt muſt be lengthened by $\frac{1}{12}$ inch on the right hand and by as much on the left, to the end that a ſteel pin put thro' the round hole in any of thoſe ſliders (*g*) may draw it

(*g*) See thoſe numbers in the lower line of Fig. 70.

it on either ſide $\frac{1}{12}$ of an inch, without moving any other ſlider.

Oppoſite to the centers of the round holes at *k*, *l*, *m*, and at the diſtance of about an inch and half from each center, are three other centers *n*, *o*, *p* upon the pin board, where two concentric circles are drawn about each, one with a radius about $\frac{3}{4}$, and the other about $\frac{1}{4}$ or $\frac{3}{10}$ of an inch. Each larger circle repreſents a braſs plate having a cylindrical neck whoſe baſe is the leſſer circle and height about a ſixth of an inch. The upper half of each neck is filed ſquare and a ſkrew hole is made in the middle of it. The round plates *n*, *o* are ſkrewed upon the ſurface of the pin board with flat headed ſkrews ſunk below the ſurface of the plates; but the plate at *p* is firſt let into the pin board as deep almoſt as the plate is thick and then is ſkrewed down.

Three ſteel pins made to fit the round holes *k*, *l*, *m* in the 1^{ft}, 5^{th} and 2^{d} ſliders, already mentioned, are riveted to the far ends of three flat draught irons *N*, *O*, *P*, and each pin is kept firm to each iron plate by a ſhoulder below and a collar above.

Cut three ſlits in the fore board at *r*, *s*, *t*, directly oppoſite to the centers *n*, *o*, *p*, and having put the ſhank *y* of the ſtraiteſt draught iron *P* thro' the ſlit *t*, and its ſteel pin into the column of holes in the ſliders at *m*, and the large cylindrical hole *P* over the neck *p* which juſt fits it; by moving the ſhank ſideways the 2^{d} ſlider which has the round hole, will be moved alone by the ſteel pin. And to keep this ſlider from being ſtirred by a like motion of thoſe above or below it, a braſs waſher, or circular ſpringing plate, whoſe diameter it equal to that of the braſs circle below, is fitted tight upon the ſquare part of the braſs neck and preſſed down upon the draught iron by a ſteel ſkrew ſkrewed into the hole *p* in the middle of the neck.

The other two draught irons *O*, *N* are made crooked to go round about the end of the bridge and the extreme pins in the pin board, and are placed in like manner as before upon the braſs circles *o*, *n*; that upon *o* having another large hole wide enough to receive the ſkrew head at *p* and to give liberty to its own angular motion about the neck *o*. The hole at *N* in the third iron plate being put upon the neck *n* and over the two former plates, has two other holes wide enough to receive the ſkrew heads at *o* and *p*, and alſo to afford room for its own angular motion about the neck *n*, the planes of theſe two iron plates being ſet off upwards with cranked necks in order to move them above the other plates.

Allowing $\frac{1}{15}$ of an inch for the motion of any ſlider, or pen of a jack, the motion of any iron ſhank at its ſlit is to $\frac{1}{15}$ of an inch, in the given ratio of *pt* to *pm*, which motion is therefore determined; and the breadth of any ſhank at the ſlit added to its motion there, gives the length of the

the ſlit, which length, if experience ſhall require it, muſt either be augmented, or blocked up a little at either end or both, for adjuſting the proper quantity of the ſlider's motion.

In the like vacant places at the baſe end of the ſliders, three other columns of holes muſt be drilled thro' them all, and the holes in the 3d, 6th and 4th ſliders remaining round (*h*) all the reſt muſt be lengthened on both ſides, on purpoſe to draw any of thoſe ſliders ſeparately (without moving the reſt) by ſteel pins fixed as before in the ends of three other draught irons, of ſimilar ſhapes to thoſe at the treble end: the ſtraiteſt being next the ſide board and ſunk down a little, the 2d going over it, and the 3d over both. Note that the draught irons move under the ſtrings of the harpſichord.

Fig. 70. Two ſliders being ſaved, two ſounds belonging to two extreme notes at each end of the two progreſſions by vths, are changed both together by the extreme knobs, rather than any two belonging to the intermediate notes. For as the extreme ſounds *Gb* and *Db*, or *B*※ and *F*※※, are ſeldomer uſed than any of the means that are out of the vulgar ſcale (*i*), if either of their uſual ſubſtitutes, *F*※ and *C*※, or *C* and *G*, which are both excluded by the principals, *Gb* and *Db*, or *B*※ and *F*※※, ſhould chance to occur and interfere with them in the ſame piece of muſic, it follows that

(*h*) See thoſe numbers in the lower line of Fig. 70.
(*i*) See the Table in Plate XIX.

that ſuch accidents will happen ſeldomer in the former than in the latter caſe.

Upon communicating this method of changing the ſounds of a harpſichord at pleaſure, to two of the moſt ingenious and learned gentlemen in this Univerſity, the Reverend Mr. *Ludlam* and the Reverend Mr. *Michel*, they encouraged me to put it in practice upon a harpſichord made on purpoſe by Mr. *Kirkman*; and were ſo kind not only to direct and aſſiſt the workmen, but alſo to improve the uſual method of drawing the ſliders in the accurate ſteady manner above deſcribed; which anſwers the deſign ſo well, that a muſician even while he is playing, can without interruption change any ſound for another which he perceives is coming into uſe: which however is ſeldom required if the ſcale be properly adjuſted before he begins to play.

It may not be amiſs to obſerve that a careful workman might ſave ſome time and labour, if inſtead of widening each hole in the ſliders ſeparately, he ſhould pierce out ſome long holes between thoſe couples of narrow ones which move the jacks; leaving a few ſlender croſs pieces here and there to add ſufficient ſtrength to the ſlider while he is working it.

SECT.

SECTION IX.

Methods of Tuning an organ and other instruments.

PRACTICAL PRINCIPLES.

I. A consonance of any two musical sounds is imperfect if it beats or undulates, and is perfect if it neither beats nor undulates.

II. Any little alteration of the interval of a perfect consonance makes it beat or undulate, slower or quicker according as the alteration is smaller or greater.

III. If the interval of a perfect consonance be a little increased, the imperfect one is said to Beat Sharp; if a little diminished, to Beat Flat (*k*).

IV. An imperfect consonance will be discovered to beat sharp, if *a very small* diminution of its interval retards the beats; or to beat flat if any diminution accelerates them.

V. The harmony of a consonance is the finest and smoothest when it neither beats nor undulates, and grows gradually coarser and rougher while the beats are gradually accelerated by very small alterations of the interval.

VI. A small alteration is sooner perceived in the rate of beating than in the harmony of a consonance, and both must be attended to in tuning an

(*k*) See Schol. 5. to Prop. xx. in the Appendix.

an inſtrument, eſpecially the harpſichord, where the beats are weak and of ſhort duration.

VII. If any imperfect conſonance be ſounded immediately after another, an attentive ear can determine very nearly whether they beat equally quick, or elſe which of them beats quicker, even without counting the beats made in a given time; eſpecially upon the organ, where the beats are ſtrong and durable at pleaſure.

VIII. If ſeveral imperfect conſonances of the ſame name, as v^ths^ for inſtance (by which the whole ſcale is uſually tuned) beat equally quick, they are not equally harmonious; to make them ſo, the higher in the ſcale ought to beat as much quicker than the lower as their baſes vibrate quicker; that is, if a v^th^ be a tone higher than another, it ſhould beat quicker in the ratio of 10 to 9 or of 9 to 8 nearly; if a III^d^ higher, in the ratio of 5 to 4; if a v^th^ higher, of 3 to 2; if an VIII^th^ higher, of 2 to 1; &c.

IX. Pl. XX. Tab. IV. V. Beginning at any note as *G* of the undermoſt progreſſion by tones, if two aſcending v^ths^ *G d*, *d a* and the deſcending VIII^th^ *a A* and the two next aſcending v^ths^ *A e*, *e b* be all made perfect, the VI^ths^ *G e*, *d b* and the x^th^ *G b* will be found to beat ſharp and ſo quick as to offend the ear by the coarſeneſs of their harmony compared with that of the v^ths^. Which ſhews that in order to make all thoſe concords and their complements to, and compounds with VIII^ths^ more equally harmonious, the intervals of of the v^{ths} and consequently of the vi^{ths} and x^{ths} must be a little diminished in the following manner.

PRECEPTS
for tuning an organ or harpsichord by estimation and judgment of the ear.

I. Pl. xx. Tab. v. Alter successively the trebles of the ascending v^{ths} *G d*, *d a* till they beat flat, the lower v^{th} very slowly and the higher a little quicker; and the descending $VIII^{th}$ *a A* being made perfect, alter the trebles of the next ascending v^{ths} *Ae*, *eb* till they beat flat *a very little* quicker than the two former *respectively*.

Then if the x^{th} *Gb*, between the base of the first and treble of the fourth v^{th}, beats sharp and about as quick as the v^{th} *G d* to the same base, those 6 notes are properly tuned for the defective scale of Organs and Harpsichords in common use.

But if that x^{th} *G b* beats sharp considerably quicker than the v^{th} *G d*, every one of those four v^{ths} *G d*, *d a*, *A e*, *e b* must be made *a very little* flatter in order to beat a *very little* quicker than before.

On the contrary, if the x^{th} *G b* beats sharp, considerably slower than the v^{th} *G d*, or not at all, or flat, those four v^{ths} must be made sharper, to beat flat a very little slower in the first case, and still slower in the second and third cases; till an equality

equality of beats of the ſaid xth and vth to the ſame baſe *G* be nearly obtained.

In like manner, the deſcending VIIIth *b B* being made perfect, let the two next aſcending v^{ths} *B f*※, *f*※ *c* ※ be made to beat flat *a very little* quicker than the v^{ths} *A e*, *e b* reſpectively, ſo as make the xth *A c*※ beat ſharp about as quick as the vth *A e* to the ſame baſe.

Laſtly make the III^{d} *e g*※ beat as quick as the vth *e b* to the ſame baſe: Becauſe the III^{d} beats juſt as quick as the xth to the ſame baſe, of neceſſity.

If we had begun at the loweſt note *E*♭, the whole ſcale might have been tuned by aſcending two v^{ths} and deſcending an VIIIth alternately; but it is better to tune the lower part of it in going backwards from *G*, aſcending by an VIIIth and deſcending by two v^{ths} alternately, as follows.

Let the aſcending VIIIth *G g* be made perfect, and by altering ſucceſſively the baſes of the deſcending v^{ths} *g c*, *c F*, make them beat flat *a very little* ſlower than the v^{ths} *a d*, *d G reſpectively*, till the xth *F a* and the vth *F c*, to the common baſe *F*, beat equally quick; and thus thoſe two v^{ths} are properly tuned, and ſo may the reſt as the notes direct.

Then tune perfect VIII^{ths} to every one of thoſe Notes.

2. Pl. xx. Tab. v. If the inſtrument has a changeable ſcale (*l*) having put out all the ſounds by their ſtops but thoſe of the common ſcale, the

(*l*) Sect. VIII. art. 11, 18, 19.

the method of tuning it is the ſame as before, obſerving only that every one of the four v^{ths}, as *Gd*, *da*, *Ae*, *eb*, muſt beat flat *a very little* quicker than before in the common defective ſcale, till every v^{th} and vi^{th} to the ſame baſe beat equally quick, the v^{ths} flat and the vi^{ths} ſharp, as *Gd* and *Ge*, *da* and *db*, *Ae* and *Af*※, &c. and theſe being ſo adjuſted, every x^{th} and iii^{d} as *Gb* and *GB*, *df*※, *Ac*※, *eg*※, will beat ſlowly flat as they ought to do.

Likewiſe in tuning backwards from *G* or *g*, the deſcending v^{ths} *gc*, *cF* muſt alſo beat flat *a very little* ſlower than *ad*, *dG reſpectively*, till every v^{th} and vi^{th} to the ſame baſe, as *cg* and *ca*, *Fc* and *Fd*, &c beat equally quick; and theſe being ſo adjuſted the x^{th} and iii^{d} as *Fa* and *FA*, &c, will beat flat very ſlowly, as they ought. Then having tuned 8^{ths} to all the ſounds of the common ſcale, change *E*♭, *B*♭, *F*, *C*, *G*, *D* and all their 8^{ths} into *D*※, *A*※, *E*※, *B*※, *F*※※, *C*※※ and their 8^{ths}, and according to the notes in Tab. $II.^{d}$ Pl. xx, proceed as before to tune the reſt of the aſcending v^{ths} *G*※ *d*※, *d*※ *a*※, &c.

Laſtly change *G*※, *C*※, *F*※, &c and their 8^{ths} into *A*♭, *D*♭, *G*♭ &c and their 8^{ths}, and according to the notes in the ſame Table proceed as before to tune the reſt of the deſcending v^{ths} *c*♭ *A*♭, *a*♭ *d*♭, *d*♭ *G*♭, &c.

But if *d*♭ be the extreme flat note in the inſtrument, alter the baſe of the v^{th} *A*♭ *e*♭ till it beats flat and juſt as faſt as the vi^{th} *A*♭ *f* beats ſharp; likewiſe

likewise alter d^b till the vth $d^b\ a^b$ beats flat and just as fast as the vith $d^b\ b^b$ beats sharp.

Then put out all the sounds but those of the common scale, to be changed again as occasions shall require; and the harmony of the whole will be much more delicate than usual in the opinion of such judges as can distinguish when harmony is fine and when it is coarse (*m*).

PROPOSITION XVIII.

To find the pitch of an organ.

THE FIRST METHOD.

By the following experiment made upon our organ at Trinity College, I found that the particles of air in the cylindrical pipe called *d*, or *de-la-sol-re* in the middle of the open diapason (*n*) made 262 complete vibrations, or returns to the places they went from, in one second of time. And this number of vibrations is what I call the Pitch of the Organ.

Having suspended a brass weight of seven pounds averdupois at one end of a copper wire, commonly used for some of the lowest notes of a harpsichord, I lapped the other end round a peg, taken from a violin, that turned stifly in a hole made in the wainscot near the organ. Then by turning the peg to and fro I lengthened or short-ened

(*m*) See Prop. xx. Schol. 1. art. 5.

(*n*) See the notation, Tab. iv. Plate xx.

ened the vibrating part of the wire, till it ſounded a double octave below the ſound of the pipe *d* abovementioned. Then having meaſured the length of the vibrating part of the wire, while ſtretched by the weight, from the loop below to the under ſide of the peg, and added to it the ſemidiameter of the peg, I cut the wire, firſt at the point of contact with the ſide of the peg, and then at the loop below; for I found no need of a bridge either above or below. And having repeated the experiment with another piece of wire taken from the ſame bunch, I found no ſenſible difference either in the length or weight of the vibrating part; the length being 35.55 inches and the weight 31 grains troy; and the ſeven pounds averdupois that ſtretched it, was equal to 49000 grains troy, allowing 7000 for each pound.

Hence by a Theorem hereafter demonſtrated (*o*), the number of ſemivibrations, forwards and backwards together, made by the wire in one ſecond was 131, and the number of ſuch vibrations made by the air in the pipe *d*, two octaves higher, was 4×131 (*p*), and ſo the number of complete vibrations (*q*) was 2×131 or 262 Q. E. I.

Scholium.

I made that experiment in the month of September at a time when the thermometer ſtood at temperate or thereabouts.

N But

(*o*) Prop. xxiv. coroll. 1, 2. (*p*) Sect. 1. art. 3 and 7.
(*q*) Sect. 1. art. 12.

But upon a cold day in November I found by a like experiment, that the ſame pipe gave but 254 complete vibrations in one ſecond; ſo that the pitch of its ſound was lower than in September by ſomething more than $\frac{1}{4}$ of a mean tone.

And upon a pretty hot day in Auguſt I collected from another experiment, that the ſame pipe gave 268 complete vibrations in a ſecond of time; which ſhews that its pitch was higher than in November by almoſt half a mean tone.

By ſome obſervations made upon the contraction and expanſion of air, from its greateſt degree of cold in our climate to its greateſt degree of heat (*r*), compared with Sir *Iſaac Newton*'s theory of the velocity of ſounds, I find alſo that the air in an organ pipe may vary the number of its vibrations made in a given time, in the ratio of 15 to 16; which anſwers to the major hemitone or about $\frac{7}{12}$ of the mean tone and agrees very well with the foregoing experiments.

Coroll. In order to know when the pitch of an organ varies, and when it returns to the ſame again, it is convenient to keep a thermometer conſtantly in the organ caſe.

(*r*) See Mr. *Cotes*'s xv^th^ Lecture upon Hydroſtat. and Pneumat. towards the end.

THE SECOND METHOD

of finding the pitch of an organ, or the number of vibrations made in a given time by any given note.

Let the notes *C*, *D*, *E*, *F*, *G*, *A*, *B*, *c*, *d*, *e*, *f*, *g*, *a*, *b*, *c'*, be in the middle of the ſcale, and from any Baſe note not higher than *D*, tune upwards three ſucceſſive perfect $\mathrm{v}^{\mathrm{ths}}$ *DA*, *Ae*, *eb*, and downwards the perfect $\mathrm{vi}^{\mathrm{th}}$ *bd*; then having counted the number of beats made in any given time by the imperfect $\mathrm{viii}^{\mathrm{th}}$ *Dd*, the number of complete vibrations made in that time by its treble *d*, will be 81 times that number of beats.

For example ſuppoſe the $\mathrm{viii}^{\mathrm{th}}$ *Dd* be found to beat 65 times in 20 ſeconds; then 81×65 or 5265 is the number of complete vibrations made by the treble *d* in 20 ſeconds; and $\frac{5265}{20}$ or 263 is the number made in one ſecond.

The time may be meaſured either by a watch that ſhews ſeconds, or a pendulum-clock, or a ſimple pendulum that vibrates forwards or backwards in one ſecond, whoſe length from the point of ſuſpenſion to the center of the bullet is 39 inches and one eighth. And the perſon that obſerves the meaſure of the time muſt give a ſtamp with his foot at the beginning and end of it, while another perſon counts the number of the beats made in that time; which number diminiſhed by one, is the number of the intervals between thoſe ſucceſſive beats and properly ſpeaking is the number required. For greater accuracy the experiment ſhould be repeated ſeveral times and a medium taken among the ſeveral reſults.

DEMONSTRATION.

For if the notes *D*, *E*, *F*, &c, ſtand for the times of the ſingle vibrations of their ſounds, we have $D : A :: 3 : 2$, $A : e :: 3 : 2$, $e : b :: 3 : 2$ and $b : d :: 3 : 5$; and by compounding thoſe ratios we have $D : d :: 81 : 40$, which ratio being reſolved into $81 : 80$ and $80 : 40$, ſhews that the VIIIth *D d* is tempered ſharp by a comma. Whence in caſ. 1. Prop XI, putting $p = 1 = q$, $n = 1$, we have $\beta = \frac{2qnM}{161p + q} = \frac{M}{81}$ and $81\beta = M$ the number of complete vibrations of the treble *d* of that VIIIth.

By *the firſt method d* was found to vibrate 262 times in one ſecond, which number being ſubſtituted for *M* ſhews that the VIIIth *Dd* in that organ made $\frac{262}{81}$ or $3\frac{11}{81}$ beats in one ſecond, which are not too quick to be counted.

Coroll. 1. Hence the numbers of vibrations made in a given time by any ſounds in the Diatonic Syſtem (*s*) are given in the given organ; being

(*s*) Sect. II^{d}. art. 1.

being reciprocally in the known ratios of their ſingle vibrations.

Coroll. 2. By diminiſhing each of the aſcending perfect v[ths] *DA*, *Ae*, *eb* by $\frac{1}{4}$ of a comma and increaſing the deſcending vi[th] *bd* by $\frac{1}{4}$ of a comma, the viii[th] *Dd* which was too ſharp by a comma, becomes perfect: which is another proof of the vulgar temperament (*t*).

THE THIRD METHOD

of finding by experiment the number of vibrations N *made by any given ſound* c *of a given organ in a known time* T.

In the ſame notation as before let *ca* be a ſharp vi[th] making β beats in the time T; make *ad*, *dG*, *GC* perfect v[ths], and *Cc* will be an imperfect viii[th]; which if ſharp and making *b* beats in the time T, will give $N = 81b + 16\beta$; or if flat, $N = 16\beta - 81b$.

For $c : a :: 5 + \frac{\beta}{N} : 3$ (*u*) and $a : d :: 2 : 3$ and $d : G :: 2 : 3$ and $G : C :: 2 : 3$. Whence by compounding all thoſe ratios, $c : C :: 40 + \frac{8\beta}{N} : 81$ and this viii[th] being ſharp and making

(*t*) Coroll. Prop. 11[d].
(*u*) Prop. xi. cor. 7. caſ. 1.

b beats in the time T, gives $C : c :: 2 : 1 - \frac{b}{N}$ (x). Wherefore $80 + \frac{16\beta}{N} = 81 - \frac{81b}{N}$ and $N = 81b + 16\beta$.

The VIIIth Cc cannot well come out flat nor can its beats be too slow and indistinct, unless those of the sharp VIth ca were too quick to be easily counted. This may be collected from *the second method*.

THE FOURTH METHOD

of finding the pitch of an organ, or N *the number of vibrations of the sound* c *in a known time* T.

The notation of the scale being still the same as in the *first method* and the octave Gg being perfect, let the vths cg, Gd, da, be made to beat flat 40, 30, 45 times respectively in a known time T; then, if ca be a sharp VIth making β beats in the time T, we have $N = 3240 + 16\beta$. But if it be a flat VIth, $N = 3240 - 16\beta$.

For $c : g :: 3 - \frac{40}{N} : 2$ (y) and $g : G :: 1 : 2$ and $G : d :: 3 - \frac{30G}{Nc} : 2$ (z) and $d : a :: 3 - \frac{45d}{Nc} : 2$ (z). Whence by compounding all these ratios,

(x) Prop. XI. cor. 7. cas. 1.
(y) Prop. XI. cor. 7. cas. 2.
(z) Prop. XI. cor. 7. and coroll. 1. to the 2d *method*.

ratios, $c : a :: 27 - \frac{1080}{N} : 16$; where if ca be a ſharp vi[th] making β beats in the time T, $c : a :: 5 + \frac{\beta}{N} : 3$ (*a*). Whence $27 - \frac{1080}{N} : 16 :: 5 + \frac{\beta}{N} : 3$ and therefore $N = 3240 + 16\beta$. But if ca be a flat vi[th], $N = 3240 - 16\beta$.

If the reſulting vi[th] ca beats ſharp and too ſlowly and therefore too indiſtinctly, or elſe too quick to be eaſily counted; make the v[th] cg beat ſlower or quicker reſpectively. Alſo if that vi[th] beats flat and conſequently too ſlowly, make the v[th] cg beat much quicker. The moſt convenient rate of beating is between 2 and 3 beats in a ſecond *.

PROPOSITION XIX.

The pitch of an organ and the temperament of the v[th] *being given, to find the numbers of beats that every* v[th] *will make in a given time.*

The muſical notes in Tab. 1 or 2, Plate xx, ſhew all the v[ths] of different names in a complete ſcale of ſounds; which v[ths] by interpoſing 8[ths] are placed

N 4

(*a*) Prop. xi. cor. 7. caſ. 1.

* In July 1751 that excellent mathematician the Reverend and Learned Mr. *Tho. Bayes* F, R, S, was pleaſed to ſend me theſe two laſt methods, in return for a method of Tuning an Organ deſcribed in ſcholium 2 to Prop. xx, which I had ſent him ſometime before.

placed at ſuch a pitch that the higher v^{ths} may not beat too quick to be counted, nor the lower too ſlow to be diſtinguiſhed. By the given temperament of the vth and the given pitch of the organ, the number β of the beats made in a given time by the vth da may be found by Prop. XI, and alſo the conſtant ratio v to 1 of the times of the ſingle vibrations of the baſe and treble of the vth.

Then if a ſeries of v^{ths} equally tempered aſcend continually from d, as in Tab. 3, the numbers of their beats made in a given time, will continually increaſe in the ratio of 1 to v (b) and therefore will be β, βv, βv^2, βv^3, βv^4, βv^5, βv^6, βv^7, βv^8, βv^9.

Now as often as any of theſe v^{ths} are depreſſed by an octave, as in the upper half of the ſcale aſcending from d in Tab. 1 or 2, ſo often muſt theſe beats be divided by 2 (b); which changes that ſeries of beats into this, β, $\frac{\beta v}{2}$, $\frac{\beta v^2}{2}$, $\frac{\beta v^3}{4}$, $\frac{\beta v^4}{8}$, $\frac{\beta v^5}{8}$, $\frac{\beta v^6}{16}$, $\frac{\beta v^7}{16}$, $\frac{\beta v^8}{32}$, $\frac{\beta v^9}{32}$.

In like manner, if the former ſeries of v^{ths} be continued downwards from d, as in Tab. 3, the numbers of their beats made in a given time will continually decreaſe in the ratio of v to 1 or of 1 to $\frac{1}{v}$, and therefore will be, $\frac{\beta}{v}$, $\frac{\beta}{v^2}$, $\frac{\beta}{v^3}$, $\frac{\beta}{v^4}$, $\frac{\beta}{v^5}$, $\frac{\beta}{v^6}$, $\frac{\beta}{v^7}$, $\frac{\beta}{v^8}$, $\frac{\beta}{v^9}$, $\frac{\beta}{v^{10}}$.

Now

(b) Prop. XI. coroll. 2.

Now as often as theſe v^{ths} are raiſed by an octave, as in the lower half of the ſcale deſcending from *d*, in Tab. 1 or 2, ſo often muſt theſe beats be multiplied by 2; which produces this ſeries of beats, $\frac{\beta}{v}, \frac{2\beta}{v^2}, \frac{2\beta}{v^3}, \frac{4\beta}{v^4}, \frac{8\beta}{v^5}, \frac{8\beta}{v^6}, \frac{16\beta}{v^7}, \frac{16\beta}{v^8}, \frac{32\beta}{v^9}, \frac{32\beta}{v^{10}}$. Q. E. I.

Scholium.

For example, be it propoſed to calculate the number of beats, in Tab. 1, Plate xx, which every v^{th} in the ſyſtem of mean tones will make in 15 ſeconds of time.

Here the temperament of the v^{th} is $\frac{1}{4}$ of a comma (c), and ſuppoſing the interval of the perfect $v^{th} = \log. \frac{3}{2} = 0.1760913$, we have the comma $c = \log. \frac{81}{80} = 0.0053950$, and $\frac{1}{4}c = 0.0013488$, and the $v^{th} - \frac{1}{4}c = 0.1747425$, which is the logarithm of the number 1.4953 or $\frac{1.4953}{1}$, that is, of the ratio of 1.4953 to 1, which in the ſolution of the problem we repreſented by v to 1.

Now when the thermometer ſtood at temperate, the pitch of our organ at Trinity College, or the number of complete vibrations made in 1 ſecond by the air in the pipe denoted by *d* in the middle of our table, was 262 (*d*).

Hence

(*c*) Prop. II^d. cor. (*d*) Prop. XVIII.

Hence to find the number of beats made in 15 ſeconds by the v^{th} above *d* when tempered flat by $\frac{1}{4}$ comma, in prop. XI we have $m:n::3:2$ or $m=3$, $n=2$ and $\frac{q}{p}c=\frac{1}{4}c$, or $q=1$, $p=4$, and ſince the baſe *d* makes 262 complete vibrations in 1 ſecond, in the given time of 15 ſeconds it will make 15×262 ſuch vibrations $=N$, and the

Abacus 1.

0. 1747425	v	N° of beats.
1. 5629841	$\beta = 36,558$	$\beta = 37$
1. 7377266	$\beta v = 54,667$	$\frac{1}{2}\beta v = 27$
1. 9124691	$\beta v^2 = 81,747$	$\frac{1}{2}\beta v^2 = 41$
2. 0872116	$\beta v^3 = 122,24$	$\frac{1}{4}\beta v^3 = 31$
2. 2619541	$\beta v^4 = 182,79$	$\frac{1}{8}\beta v^4 = 23$
2. 4366966	$\beta v^5 = 273,34$	$\frac{1}{8}\beta v^5 = 34$
2. 6114391	$\beta v^6 = 408,73$	$\frac{1}{16}\beta v^6 = 26$
2. 7861816	$\beta v^7 = 611,20$	$\frac{1}{16}\beta v^7 = 38$
2. 9609241	$\beta v^8 = 913,95$	$\frac{1}{32}\beta v^8 = 29$
3. 1356666	$\beta v^9 = 1366,7$	$\frac{1}{32}\beta v^9 = 43$

the number of beats made in that time, by caſ. 2, is $\frac{2qmN}{161p+q} = \frac{2\times3\times15\times262}{161\times4+1} = \frac{1572}{43} = \beta$, whoſe logarithm is 1.5629841; to which adding continually the log. of v, we get the logarithms of βv,

Abacus 2.

$\bar{1}.8252575$	$\frac{1}{v}$		Nº of beats
1.5629841	β	$=36,558$	$\beta = 37$
1.3882416	$\frac{\beta}{v}$	$=24,448$	$\frac{\beta}{v} = 24$
1.2134991	$\frac{\beta}{v^2}$	$=16,349$	$\frac{2\beta}{v^2} = 33$
1.0387566	$\frac{\beta}{v^3}$	$=10,933$	$\frac{2\beta}{v^3} = 22$
0.8640141	$\frac{\beta}{v^4}$	$=7,3116$	$\frac{4\beta}{v^4} = 29$
0.6892716	$\frac{\beta}{v^5}$	$=4,8896$	$\frac{8\beta}{v^5} = 39$
0.5145291	$\frac{\beta}{v^6}$	$=3,2699$	$\frac{8\beta}{v^6} = 26$
0.3397866	$\frac{\beta}{v^7}$	$=2,1867$	$\frac{16\beta}{v^7} = 35$
0.1650441	$\frac{\beta}{v^8}$	$=1,4623$	$\frac{16\beta}{v^8} = 23$
$\bar{1}.9903016$	$\frac{\beta}{v^9}$	$=0,9779$	$\frac{32\beta}{v^9} = 31$
$\bar{1}.8155591$	$\frac{\beta}{v^{10}}$	$=0,6540$	$\frac{32\beta}{v^{10}} = 21$

βv, βv^2, βv^3 &c, as in the first Abacus, and thence the corresponding numbers, which divided by the proper powers of 2, as directed in the solution of the problem, give the ascending half of the set of beats opposite to the pitch 262 in Tab. 1.

The log. of v subtracted from o gives the log. of $\frac{1}{v}$, which log. continually added to the log. of β, gives the logarithms of $\frac{\beta}{v}$, $\frac{\beta}{v^2}$, $\frac{\beta}{v^3}$, &c as in the 2d Abacus. And these logarithms give the numbers themselves, which multiplied by the proper powers of 2, as above directed, give the descending half of the same set of beats opposite to the pitch 262 in Tab. 1.

The superior sets of beats are designed for tuning the same or different organs, when their pitch is higher than this by 1, 2, 3 or 4 quarter-tones, as noted at the beginning of the table, and may be found by the continual addition of the logarithm of a quarter-tone to the logarithms in each Abacus; and the first inferior set of beats may be found by the subtraction of the log. of a quarter-tone from the said set of logarithms, or by the addition of its arithmetical complement: remembering to divide and multiply the corresponding numbers by the same powers of 2 as before in each Abacus.

And as the $\frac{1}{4}$ tone is $\frac{1}{8}$ of the IIId, its logarithm is $\frac{1}{8}\log.\frac{5}{4} = 0.0121138$, which continually

nually added to the logarithm of 262, gives the ſucceſſive logarithms of the higher pitches 269, 277, &c, in the firſt column of the table, over againſt the correſponding ſuperior ſets of beats; and ſubtracted from the ſame logarithm of 262, gives the log. of the loweſt pitch 255, overagainſt the loweſt ſet of beats.

From the given temperament of the ſyſtem of equal harmony (*e*) the beats of all the v^{ths} may be calculated by the ſame method; and will be found as in Tab. II[d]. Plate XX.

Coroll. 1. Suppoſing the middlemoſt notes *d*, in the firſt and ſecond tables, to be uniſons, the numbers of the beats in a given time, of any two correſponding v[ths] are very nearly in the given ratio of their temperaments $\frac{9}{36}c$ and $\frac{10}{36}c$ (*f*), or as 9 to 10. For the beats would be in that ratio if their ſeveral baſe notes were exactly uniſons (*g*); and the difference of their pitches at the diſtance of the tenth v[th] from the middle note *d*, is but ten times the difference of the temperaments, or $\frac{5}{18}c$; which produces the difference of but 1 beat in 290 in the extreme v[ths] in the tables, and leſs in the reſt in proportion to their diſtances from the note *d* (*h*).

Coroll. 2. In the ſyſtem of equal harmony the ratio of the numbers of beats of the III[d] and v[th] to

(*e*) Prop. XVI, Schol. 2. art. 10.
(*f*) Prop. 2. cor. and prop. XVI. ſchol. 2. art. 10.
(*g*) Prop. XI. coroll. 4. and ſchol. 1.
(*h*) Cor. 4. lemm. to prop. IX. and cor. 2. prop. XI.

to the same base, is 2 to 3 in a given time; as being compounded of the ratio of their temperaments $\frac{2}{18}$ and $\frac{5}{18}c$, and of the major terms of their perfect ratios 5 : 4 and 3 : 2 (*i*).

Coroll. 3. For the same reason the beats of the v^th^ and vi^th^ to the same base are isochronous in the system of equal harmony, whereas in that of mean tones they are in the ratio of 3 to 5 in a given time.

PROPOSITION XX.

To tune any given organ by a given table of beats.

Having found the pitch of the organ by any of the methods in Prop. XVIII, look for the nearest to it in the first column of Table I or II, Plate XX, and overagainst it is the *Proper Set of Beats* for tuning the given organ. If the weather be considerably hotter or colder at the time of tuning than it was when the pitch was found, allowance must be made in the number of vibrations denoting the pitch, by the schol. to prop. XVIII.

Then flatten the treble *a* of the perfect v^th^ above *d* (*k*) more or less, till the number of its beats, made in 15 seconds (*l*), agrees with the tabular

(*i*) Coroll. 3. prop. XI and schol. I.

(*k*) See the notation tab. IV. plate XX.

(*l*) To be measured as directed in the Second Method in prop. XVIII. But in this case count one beat more than the tabular number, as properly signifying the number of intervals between the successive beats.

tabular number placed over that vth in the proper ſet.

From the treble of that vth *da* tune downwards the octave *aA*, ſo as to be quite free from beats, and repeat the like operation upon the next aſcending vth *Ae*, and the like again upon the next till you have tuned all the ſharp notes in the ſcale of your organ.

Then going backwards from *d*, ſharpen the baſe *G* of the perfect vth below *d* more or leſs till the number of its beats, made in 15 ſeconds, agrees with the tabular number placed over this vth in the proper ſet.

From the baſe of the vth *dG*, tune upwards the octave *Gg* and repeat the like operation upon the next deſcending vth *gc*, and the like again upon the next, till you have tuned all the flat notes in the ſcale of your organ.

This being done, let all the other ſounds be made octaves to theſe, and the ſcale will be exactly tuned according to the temperament in the given table; that is, all the v^{ths} will be equally tempered, and conſequently equally harmonious (*m*), and ſo will all the VI^{ths}, and every other ſet of concords of the ſame name, which anſwers the deſign of tuning by a table of beats.

If you chuſe to tune the organ according to the *Hugenian* ſyſtem, the ſet of beats in Tab. 1, next below that which anſwers to the pitch, found by any of the foregoing methods, will ſerve your purpoſe.

For

(*m*) Prop. XII coroll.

For the *Hugenian* v[th], having its temperament $-\frac{1}{4}c + \frac{1}{110}c$ smaller than $-\frac{1}{4}c$, in the ratio of 53 to 55 (*n*), beats slower than the tabular v[th] in that proportion, which is but very little slower than it would do, if its pitch were depressed by $\frac{1}{4}$ of a mean tone. Q. E. F.

Scholium I.

1. Since our organ at Trinity College was new voiced, and by altering the disposition of the keys was depressed a tone lower, and thereby reduced to the Roman pitch, as I judge by its agreement with that of the pitch pipes made about the year 1720; by the help of such a pipe one may know by how many quarter-tones the pitch of any other organ is higher than that of ours, and thus (without any of the methods in Prop. XVIII) determine the proper set of beats for tuning it (*o*).

2. At a time when the thermometer stood at Temperate, as it did also when the pitch of our organ was found to be 262, I assisted at the tuning of the v[ths] of the open Diapason by the set of beats opposite to that pitch in Tab. 1, and upon examining the III[ds] and X[ths] I found them all perfect: a manifest proof of the theory of beats and of the certainty of success in tuning by it.

3. At that time the whole organ was tuned to the open diapason, and is now universally allowed to

(*n*) Prop. XVII. Schol. Tab. 2.
(*o*) See column 2, Tab. 1, 2, plate XX.

to be much more harmonious than before, when the major thirds were much ſharper than perfect ones; and its harmony, I doubt not, is ſtill improveable by making them flatter than perfect, according to the ſyſtem of equal harmony. But at that time I had not finiſhed the calculation of it, and to repeat the tuning of the organ over again would be troubleſome and improper at the preſent ſeaſon, when cold and damp weather is coming on very faſt.

4. For the propereſt times for tuning the Diapaſon of an organ ſeem to be from the latter end of Auguſt to the middle of October, when the air being dry, temperate and quiet, will keep nearer to the ſame degree of elaſticity for a given time. Becauſe a very ſmall alteration in the warmth of moiſt air will ſuddenly and ſenſibly alter its elaſtic force and thereby the pitch of the pipes before the whole ſtop can be accurately tuned.

For that reaſon conſtant care muſt be taken not to heat the pipes by touching them oftener than is needful; nor to ſtay too long at a time in the organ caſe; nor to tune early in the morning, but rather towards the evening, when the air is drier and its declining warmth is kept at a ſtay by the warmth of the perſons about the organ.

But theſe and the like cautions may ſooner be learned by a little practice than by any deſcription, and if not altogether neceſſary, will however contribute to the accuracy of tuning by ſo nice a method which is plainly capable of any desired degree of exactness provided the blast of the bellows be uniform.

5. After tuning an organ according to any new system whatever, we must be cautious of judging too hastily of it. Some musicians here who had constantly been used to major thirds and consequently major sixths tuned very sharp, could not well relish the finer harmony of perfect thirds and better sixths in the organ newly tuned, till after a little use they became better satisfied with it, and after a longer use they could not bear the coarse harmony of other organs tuned in the usual manner.

It is therefore necessary to have equal experience in different objects of sense, in order to judge impartially, which of the two is more grateful than the other, as is evident in almost every thing to which we are more or less habituated.

6. If a machine were contrived, as it easily might, to beat like a clock or watch, any given number of times in 15 seconds, between 20 and 56 or thereabouts; by setting it to beat according to any given number in the table for tuning an organ, and by comparing its beats with those of the corresponding vth, the ear would determine immediately and exactly enough, whether they were isochronous or not; and thus a harpsichord might be tuned almost to the same exactness as an organ; and the tuning of an organ might be performed much quicker by the help of such a machine than by counting the beats as above.

In

In the following method after 3 or 4 fifths are tuned by a little table of beats, the organ itſelf does the office of ſuch a machine in all the reſt.

Scholium 2.

To tune any given organ by iſochronous beats of different concords.

1. The ſounds in the middle of the ſcale being called *C D E F G A B c d e f g a b c'*, make the VIIIth *G g* quite perfect, and let the V^{ths} *cg*, *Gd*, *da* be made to beat flat 38, 28, 42 times reſpectively in 15 ſeconds of time.

N° of beats of the VIth	The ſyſtem of Equal Harmony. Tab. 1.			A proper ſyſtem for defective ſcales. Tab. 2.			
	The numbers of beats of the V^{ths}.						
c a	*cg*	*Gd*	*da*	*cg*	*Gd*	*da*	*Ae*
56	41.	31−	46−	32+	24−	36−	27−
50	40.	30−	45	31+	23+	35−	26+
44	39.	29+	44−	31−	23−	34+	25+
38	38.	28+	42+	30−	22+	33+	25−
32	37.	28−	41+	29−	22−	32+	24+
26	36.	27−	40+	28+	21+	31+	24−
20	35.	26+	39+	27+	20+	31−	23−

Then having counted the number of beats made in 15 ſeconds by the reſulting VIth *ca*, look for it or the neareſt number to it in the column of the beats of that VIth placed before the tables. The

 numbers

numbers oppofite to it in each table are the Proper Set of Beats, which the faid v^ths^ ought to make in the given organ, when tuned according to the fyftem mentioned in the title of each table.

2. For example, fuppofe the refulting vi^th^ fhould beat 48 times in 15 feconds, the neareft number to it in the firft column is 50 and oppofite to it in Tab. 1, are 40, 30, 45, the proper fet of beats of thofe v^ths^ *cg*, *Gd*, *da* for caufing the v^th^ and vi^th^, *cg* and *ca*, to beat equally quick as they ought to do in the fyftem of equal harmony (*p*).

Likewife in Tab. 2, oppofite to the faid number 50 are 31, 23, 35, 26, the proper fet of beats of the v^ths^ *cg*, *Gd*, *da*, *Ae*, for caufing the v^th^ and iii^d^, *cg*, *ce*, to beat equally quick, which property makes a very proper fyftem for the defective fcales of organs and harpfichords in prefent ufe.

3. Alter then the trebles of the v^ths^ *cg*, *Gd*, *da* till they beat flat 40, 30, 45 times refpectively in 15 feconds, remembering by the way to make *gG* a perfect viii^th^. Then upon founding the v^th^ and vi^th^ *cg*, *ca*, immediately after each other, the ear will judge near enough whether they beat equally quick as they ought.

4. But if greater certainty be defired, count the beats of the vi^th^ *ca* in 15 feconds and if their number be 40 as that of the v^th^ *cg* was, or differs from it by no more than 3, over or under, you have found the proper fet; but if the beats of the vi^th^

(*p*) Prop. xix. fchol. coroll. 3.

VIth differ from 40 by more than 3 and leſs than 9, over or under, the next higher or lower ſet reſpectively, is the proper ſet. Becauſe an alteration of ſix beats of the VIth anſwers in time to an alteration of but one beat of the Vth to the ſame baſe, as appears by the differences of the numbers in the firſt and ſecond columns. For which reaſon, if in any experiment the beats of the VIth *ca* ſhould come out exactly in the middle between any two numbers in the 1^{ſt} column, you may take either of the ſets oppoſite to them for the proper ſet; the error from the truth being but half a beat, that is, half the interval of the ſucceſſive beats of the Vth *cg*, which half cannot eaſily be meaſured. Yet the choice might be determined by altering the pitch of the ſound *c* a very little and repeating the experiment.

5. The proper ſet for a given organ being once found, the 1^{ſt} experiment need never be repeated afterwards. For whatever be the proper ſet in temperate weather, the next above it will be the proper ſet in hot, and the next below it in cold weather (*q*). At moſt, the ſet ſo determined need only be verified as in Article 3 or 4 whenever the organ wants to be retuned.

6. After the V^{ths} *cg*, *Gd*, *da* had been made to beat 38, 28, 42 times in 15 ſeconds in the 1^{ſt} experiment, if the beats of the VIth *ca* had come out too quick, or too ſlow and indiſtinct to

(*q*) Prop. XVIII. ſcholium.

be eaſily counted, (which however cannot happen unleſs the pitch of the organ be immoderately higher or lower than uſual,) in the firſt caſe uſe 13 ſeconds, and in the ſecond 17 inſtead of 15; and upon repeating the experiment the beats of the vi$^{\text{th}}$ *ca* will come out within the limits 20 and 56, of the numbers in the firſt column and point to the proper ſet.

7. Laſtly, if the beats of any v$^{\text{th}}$ cannot ſoon be adjuſted to the tabular number, which ſometimes happens, and that number has the ſign + after it, the exceſs of a beat may be diſpenſed with as being leſs erroneous in that caſe than the defect of a beat; and on the contrary, if the tabular number has the ſign – after it.

8. Pl. xxv. Fig. 65. Now if you chuſe to tune the organ to the ſyſtem of equal harmony, which being the moſt harmonious is the propereſt for a changeable ſcale (*r*), in aſcending from the ſounds, *c*, *G*, *d*, *a* tuned by the proper ſet of beats, make the vi$^{\text{th}}$ *Ge* beat ſharp and juſt as quick as the given v$^{\text{th}}$ *Gd*, and do the like for every vi$^{\text{th}}$ in the order annexed.

V	VI	VIII
Gd,	*Ge*	
da,	*db*	& tuning *aA*
Ae,	*Af*※	
eb,	*ec*※	& tuning *bB*
Bf※,	*Bg*※	
&c,	&c	&c.

Likewiſe in deſcending from the ſaid given ſounds

(*r*) Sect. viii art. 11$^{\text{th}}$.

ſounds c, G, d, a, alter the common baſe F of the vth and vith, Fc and Fd, till they beat equally quick, the vth flat, and the vith ſharp; and repeat the like practice in the order annexed.

v	vi	viii
Fc,	Fd & tuning	Ff
$B^{b}f$,	$B^{b}g$	
$E^{b}B^{b}$,	$E^{b}c$ & tuning	$E^{b}e^{b}$
$A^{b}E^{b}$,	$A^{b}f$	
&c,	&c	&c.

If this ſcale aſcend ſo high as to cauſe a vth and vith, as eb and ec* for inſtance, to beat too quick, tune downwards the two viii^{ths} eE and bB and leaving out the uppermoſt row of the ſharp notes, proceed with the lower rows. And do the contrary in the deſcending part of the ſcale, leaving out the undermoſt flat notes where you find they beat too ſlow and deſcending by the higher notes.

9. But till Inſtruments are made with a changeable ſcale, it is more proper to tune the defective ſcale in preſent uſe by making every vth and iii^{d}, to the ſame baſe, beat equally quick, the former flat and the latter ſharp.

Make the v^{ths} cg, Gd, da, Ae beat flat 31, 23, 35, 26, times reſpectively, this being the proper ſet found by Art. 1, 2, for the given organ, and the iii^{d} ce will beat ſharp equally with the vth cg.

Plate xxv. Fig. 65. Then in aſcending from the ſounds *c*, *g*, *G*, *d*, *a*, *A*, *e* ſo tuned, make the next III^d^ *GB* beat ſharp and equally with the given v^th^ *Gd*, and do the like for the reſt in the order annexed.

v	III	VIII
Gd,	*GB* and tuning	*Bb*
da,	*df*※	
Ae,	*Ac*※	
eb,	*eg*※	

Likewiſe in deſcending from the ſame given ſounds alter the baſe *F* of the III^d^ *FA* till it beats ſharp and equally with the given v^th^ *Fc* to the ſame baſe, and do the like for the reſt in the order annexed.

v	III	VIII
Fc,	*FA*, & tuning	*Ff*
B^b^f,	*B^b^d*	
*E^b^B^b^*,	*E^b^G*.	

DEMONSTRATION.

Pl. xxiv. Fig. *A*, which is deſcribed in Prop. III and cor. 1. 2. The beats of any given v^th^ and VI^th^ to the ſame baſe will be iſochronous, as they ought to be in the ſyſtem of equal harmony, when their temperaments are as 5 to 3 (*t*). Whence by the coroll. to prop. IV, Gr is $= \frac{5}{18}c$ and is the flat temperament of all the v^ths^ (*u*) and $As = \frac{3}{18}c$ is the ſharp temperament of the reſulting VI^ths^.

In Tab. 1. column 1 the aſcending numbers 35, 36, 37, 38, &c are aſſumed for the beats to be made in a given time by the given v^th^ *cg* in organs

(*t*) Prop. XI. coroll. 3 and ſchol. 1.
(*u*) Prop. XII. coroll.

organs of different pitches (*x*) and from these and the given temperament $\frac{5}{18}c$, the beats of the other v^{ths} Gd, da in col. 2. 3 of that table are computed by the method in Prop. XIX.

When the vth and vith cg and ca, have any other temperaments, as $G\rho$ and $A\sigma$, let a and b be the respective numbers of their beats made in any known time, and β their common number made in that time when their temperaments were $Gr = \frac{5}{18}c$ and $As = \frac{3}{18}c$ as before. Then supposing the difference $r\varrho = \frac{d}{18}c$, by the similar triangles $Or\varrho$, $Os\sigma$, we have $s\sigma = \frac{3d}{18}c$. And since the numbers of beats made in equal times by a given consonance differently tempered are proportional to its temperaments (*y*), we have $\beta : a :: Gr : G\varrho :: 5 : 5 - d$, whence $d = \frac{5\beta - 5a}{\beta}$; likewise $\beta : b :: As : A\sigma :: 3 : 3 + 3d$, whence again $d = \frac{b-3}{\beta}$. Therefore $5\beta - 5a = b - \beta$ and $\beta = a + \frac{b-a}{6}$.

In the method above for finding the proper set of beats we assumed $a = 38$ and in the example we supposed $b = 48$, whence $\beta = 38 + \frac{48-38}{6} = 39\frac{2}{3}$, the nearest to which in column I tab. I being 40, gives the proper set, 40, 30$\frac{1}{2}$, 45$\frac{1}{2}$.

After

(*x*) Prop. XI. cor. 2.
(*y*) Prop. XI. cor. 4. and schol. I.

After this manner I found the proper ſet for making the vth and vith, *cg*, *ca*, beat equally quick in our organ at Trinity College, to be 36, 27–, 40 +, in a warm ſeaſon, after the pitch of it had been depreſſed a tone lower, to the Roman pitch, only by changing the places of the keys. Conſequently the proper ſet for the former pitch would have been 40, 30–, 45–, becauſe the ratio of 36 to 40 or 9 to 10 belongs to the tone nearly. For this reaſon in the method above I choſe to begin the experiment with the intermediate ſet 38. 28 +, 42 +, and adapted to it the column of beats of the vith *ca* in the following manner.

When $a = 38$, $\beta = 38 + \frac{1}{6}\overline{b-38}$, whence it appears if $b = 38$, that $\beta = 38$; if b exceeds 38, that β exceeds 38 by $\frac{1}{6}$ of that exceſs; if b be deficient from 38, that β is deficient from 38 by $\frac{1}{6}$ of the defect. So that the common difference of the values of β or beats of the vth *cg* being unity in col. 1. Tab. 1, the common difference of the correſpondent values of b or beats of the vith *ca* muſt be ſix, as in the column prefixed to the Table.

If $b = 0$, that is, if the vith *ca* ſhould come out perfect, we have $\beta = 38 - 6\frac{1}{3} = 31\frac{2}{3}$, which in col. 1. Tab. 1 continued downwards would give the pitch and the proper ſet for an organ about a tone lower than ours, tho' lower than ordinary;

ordinary; becauſe the ratio of 36 to $31\frac{2}{3}$ or 32, is 9 to 8 belonging to the tone major. Therefore in the experiment above for finding the proper ſet, the VI^th^ ca cannot beat flat upon any organ now in uſe: and if its ſharp beats come out too ſlow or too quick, a remedy has been given above in art. 6. The reaſon of it is this; if either of the temperaments $G\varrho$, $A\sigma$, of the v^th^ and VI^th^ be increaſed, the other will decreaſe, and accordingly the beats of the former conſonance will be accelerated while thoſe of the latter are retarded. So that when the beats of the VI^th^ come out too quick to be eaſily counted, they ſhew that thoſe of the v^th^ were too ſlow at firſt and therefore muſt be accelerated, or the ſame number of them muſt be made in a leſs time: and on the contrary, when the beats of the VI^th^ come out too ſlow.

In Tab. 2, where the major III^d^, in order to leſſen the dieſis (z), is deſigned to beat ſharp and as quick as the v^th^ beats flat, the ratio of the temperaments $G\rho$, $E\tau$, of the v^th^ and III^d^ muſt be 5 to 3 (a), and the temperer $O\,\varrho\,\sigma\,\tau$ muſt be within the angle EOG. Hence $G\varrho = \frac{5}{23}c$ (b). But in Tab. 1 we had $Gr = \frac{5}{18}c$, and ſo the ratio of the beats made in a given time by the correſpondent

(z) Sect. VIII. art. 2. note u.

(a) Prop. XI. cor. 3. ſchol. 1.

(b) Prop. V. cor. caſ. 2, where $\frac{r}{t}$ may have any value, becauſe the ſecond condition of the propoſition is not here required.

ſpondent v^ths in Tab. 1 and 2, is Gr to $G\varrho$ or 23 to 18 (c).

In like manner a table of beats for any other ſyſtem might be computed and ſubjoined to Tab 1, in which the proper ſet may be found as before in the 1ſt experiment. Another ſyſtem might be tuned by iſochronous beats of the III^d and VI^th, but it differs ſo little from that of equal harmony that the mention of it is ſufficient.

Corollary. Hence we have the number of vibrations made in a given time by any given ſound of a given organ. For the number of complete vibrations made in a given time by the ſound c is the product of this conſtant number $96,7\frac{2}{3}$ multiplied by the number of beats made in that time by the v^th cg when it beats equally with the VI^th ca, to be found by the method above deſcribed.

Thus if that number of beats be 36 in 15 ſeconds, in this time the ſound c makes $36 \times 96,766$ &c $= 3484$ complete vibrations, that is $232\frac{1}{5}$ vibrations in one ſecond; and therefore the ſound d, which is almoſt a mean tone higher than c, makes 260 ſuch vibrations in one ſecond, which agrees with the experiment made with the braſs wire in prop. XVIII.

For the temperament of the v^th cg, when it beats flat and equally with the VI^th ca, was found to be $\frac{5}{18}c, = \frac{q}{p}c$ in prop. XI, where in caſ. 2, $\beta =$

2 q m N

(c) Prop. XI, coroll. 4.

$\frac{2qmN}{161p+q}$, or $N = \frac{161p+q}{2qm}\beta = \frac{161 \times 18 + 5}{10 \times 3}\beta = 96,7\frac{2}{3} \times \beta$, the number of vibrations of the ſound *c*.

Scholium 3.

If we could meaſure any given part of a ſecond of time more readily and exactly by any other means than by the beats themſelves, a ſingle ſet of beats for a given ſyſtem, as 38, 28, 42, &c would alone be ſufficient for tuning any given organ according to the given ſyſtem.

For, by the method above, having found β the proper number of beats which the vth *cg* ought to make in a given time as 15 ſeconds, the time *t*, in which it will make the number 38 in the given ſet, is to 15 ſeconds, as 38 is to β; and that time ſo determined is the Proper Time in which all the other numbers of beats in the given ſet ought to be made by the other v^{ths} in the ſaid organ. But unleſs that time *t*, which will generally contain ſome fraction of a ſecond, could be readily and accurately meaſured, this method will with equal expedition be leſs accurate, or with equal accuracy will be leſs expeditious than the former.

For if inſtead of the mixt number *t* we uſe the neareſt whole number of ſeconds for the proper time, the limit of the error will be half a ſecond; whereas in uſing the Proper Set for any given time, the error is but half the interval of the ſucceſſive beats of the vth *cg*, which is two

or

or three times ſmaller than half a ſecond, becauſe the number of its beats in col. 1. Tab. 1, is always between two and three times greater than 15, the number of ſeconds in which the beats are made. And the larger error cannot eaſily be reduced to an equality with the ſmaller, unleſs by a ſet of beats whoſe numbers are between 2 and 3 times larger than thoſe in the Tables, which would proportionally increaſe the time and trouble of counting them. For inſtance, inſtead of counting 38, 28, 42 beats in 15 ſeconds, we muſt count 96, 72, 107, in 38 ſeconds. Becauſe 15, 38, 96 are continual proportionals nearly.

As the known method of tuning an inſtrument by the help of a monochord is eaſier than any other to leſs skilful ears, and pretty exact too if the *apparatus* to the monochord be well contrived, it may not be amiſs to ſhew the manner of dividing it according to any propoſed temperament of the ſcale.

PROPOSITION XXI.

To find the parts of a given monochord, whoſe vibrations ſhall give all the ſounds in an octave of any propoſed tempered ſyſtem.

Let the ſyſtem of equal harmony be propoſed, and let the ſeveral parts of the monochord be meaſured from either end of it, and be to the whole,

whole, in the ratios of the ſeveral numbers in the 3ᵈ column of the following table, to 100000; I ſay the vibrations of the parts ſo found, and of the whole, will give all the ſounds in an octave of the propoſed ſyſtem, as denoted in the firſt column of the table. Q. E. I.

For in the ſcholium to prop. XVII we had $2T = 0.09631.05650$ and $2L = 0.06025.35832$; whence we have

$$\begin{aligned} T &= 0.04815.52825 \\ L &= \underline{0.03012.67916} \\ T - L = l &= 0.01802.84909 \\ L - l = d &= 0.01209.83007 \end{aligned}$$

From theſe logarithms of the tone, limma major and minor and the dieſis, and from the logarithm 4.69897.00043 of the number 50000, the uppermoſt in the table, all the logarithms below it will be found by the following additions: where the muſical notes in column 1 are ſuppoſed alſo to repreſent the logarithms over againſt them, till you come down to C, which comes out 5.00000.00000 and ſhews that the

$$\begin{array}{lll} c + d = B^{*}, & c + l = c^{b}, & c + L = B \\ B + l = B^{b}, & B + L = A^{*}, & B + T = A \\ A + l = A^{b}, & A + L = G^{*}, & A + T = G \\ G + l = G^{b}, & G + L = F^{*}, & G + T = F \\ F + d = E^{*}, & F + l = F^{b}, & F + L = E \\ \&c & \&c & \&c \end{array}$$

logarithms

logarithms of the principal notes B, A, G, F, E, D are right; and those of the secondary notes will be right too, if the operations in the addition be right.

The corresponding numbers in column 3, which may be found by the tables of logarithms, shew the required parts of the monochord; as a very little reflection will satisfy any one that understands the common properties of logarithms, and attends to the intervals of an octave in Fig. 49 described in Sect. VIII, but not divided as there into 50 equal parts, which is only an approximation to the system proposed. Q. E. D.

Scholium.

The numbers in the 4th and 5th columns of the table shew the parts of a monochord, whose vibrations will give the sounds of the opposite notes in the system of mean tones and that of Mr. *Huygens*, who has shewn how to find the last column of numbers in his *Harmonic Cycle*. And as all the measures in the 3 systems may be taken and marked upon the sounding board of the same monochord, the different effects of those systems upon the ear, may be easily tried and compared together, provided the tone of the monochord be good and the divisions accurate, and the moveable bridge does not strain it in one place more than in another.

SECTION

facing p. 224.

The division of a Monochord.

1	2	3	4	5
c	4. 69897. 00043	50000	50000	50000
B※	4. 71106. 83050	51412	51200	51131
cb	4. 71699. 84952	52119	52245	52278
B	4. 72909. 67959	53592	53499	53469
B*b*	4. 74712. 52868	55863	55902	55914
A※	4. 75922. 35875	57441	57243	57179
A	4. 77725. 20784	59876	59814	59794
A*b*	4. 79527. 05693	62412	62500	62528
G※	4. 80737. 88700	64177	64000	63942
G	4. 82540. 73609	66897	66874	66866
G*b*	4. 84343. 58518	69733	69877	69924
F※	4. 85553. 41525	71702	71554	71506
F	4. 87356. 26434	74742	74767	74776
E※	4. 88566. 09441	76853	76562	76467
F*b*	4. 89159. 11343	77910	78125	78196
E	4. 90368. 94350	80110	80000	79964
E*b*	4. 92171. 79259	83506	83593	83621
D※	4. 93381. 62266	85865	85599	85512
D	4. 95184. 47175	89504	89443	89422
D*b*	4. 96987. 32084	93298	93459	93512
C※	4. 98197. 15091	95934	95702	95627
C	5. 00000. 00000	100000	100000	100000
	Syftem of	equal harmony	mean tones	Mr. *Huygens.*

SECTION X.

Of occaſional temperaments uſed in concerts well performed upon perfect inſtruments.

By a perfect inſtrument I mean a voice, violin or violoncello, &c, with which a good muſician can perfectly expreſs any ſound which his ear requires.

PROPOSITION XXII.

The ſeveral Parts of a concert well performed upon perfect inſtruments, do not move exactly by the given intervals of any one ſyſtem whatever, but only pretty nearly, and ſo as to make perfect harmony as near as poſſible.

For inſtance, if the baſe be ſuppoſed to move by the beſt ſyſtem of perfect intervals (*d*), the other part or parts cannot conſtantly move by it too, without making ſome of the concords imperfect by a comma (*e*), which would grievouſly offend the muſicians (*f*). Conſequently if they are pleaſed, thoſe intervals are occaſionally tem-

P pered

(*d*) See Sect. II. art. I.
(*e*) Sect. IV. Prop. I. and coroll.
(*f*) Sect. IV. art. 9.

pered by the upper part or parts, which therefore do not move by the ſame intervals which the baſe is ſuppoſed to move by.

Likewiſe if the baſe be ſuppoſed to move by the ſyſtem of mean tones and limmas (*g*), the other part or parts cannot conſtantly do ſo too, without making about two thirds of all the concords imperfect by a quarter of a comma (*h*). But whenever concords are held out by good muſicians, they ſeem to me to be always perfect. And if ſo, the upper part or parts cannot move by the ſyſtem of mean tones, which the baſe is ſuppoſed to move by. And the argument is the ſame if the baſe be ſuppoſed to move by any other ſyſtem of tempered intervals: and that it cannot conſtantly move by perfect ones, I ſhall ſhew in the next ſcholium.

What has been ſaid of perfect and imperfect concords, is applicable to diſcords too, a good ear being critical in both. Now the reaſon why the beſt muſicians acquire a habit of making perfect harmony, as near as poſſible, is plainly this. When the harmony is made perfect they are pleaſed and ſatisfied, though the ſeveral parts do not move by perfect intervals. For the paſſing from one ſound to the next, whether by a perfect or an imperfect interval, being nearly inſtantaneous, cannot much offend the muſician. But the ſucceeding conſonance is long enough held out to give him pleaſure or pain according as he makes it perfect or imperfect. Q. E. D.

Coroll.

(*g*) Prop. II. (*h*) Prop. III. coroll. 3.

Coroll. 1. *Cæteris paribus* the harmony will be the ſame whatever be the ſyſtem which the baſe moves by; but the ſum of the occaſional temperaments will be the leaſt poſſible, if it moves by the ſyſtem of mean tones and limmas (*i*), and but very little bigger, if it moves by the ſcale of equal harmony (*k*).

Coroll. 2. The propoſition holds true though one of the inſtruments be imperfect, as when the thorough baſe is played upon an organ or harpſichord: becauſe the performers of the upper parts are more attentive to make perfect harmony with the baſe notes, than with the chords to them. Conſequently thoſe parts do not move by the tempered ſyſtem of the thorough baſe.

Coroll, 3. *Cæteris paribus* the ſame piece of muſic well performed upon perfect inſtruments, is more agreeable than it would be if it were as well performed upon imperfect ones, as an organ, &c.

For nothing gives greater offence to the hearer, though ignorant of the cauſe of it, than thoſe rapid rattling beats of high and loud ſounds, which make imperfect conſonances with one another. And yet a few ſlow beats, like the ſlow undulations of a cloſe ſhake now and then introduced, are far from being diſagreeable.

Coroll. 4. Therefore the harmony of a concert will be ſmoother and diſtincter, and gene-

 rally

(*i*) Prop. III. coroll. 10.

(*k*) Schol. to prop. xx. art. 6. in the Appendix. and prop. 41. coroll. 4.

rally more pleaſing, for taking the chords of the thorough baſe as near as can be to the baſe notes, and no more of them than are neceſſary, and theſe few upon the ſofter and ſimpler ſtops of an organ.

Becauſe the beats will then be fewer, ſlower and ſofter, and ſo the voices and other inſtruments will appear to greater advantage.

Coroll. 5. It appears alſo from the reaſons above, that no voice part ought ever to be played on the organ, unleſs to aſſiſt an imperfect ſinger, and keep him from making worſe concords with the baſe and other parts than the organ it ſelf does.

Scholium.

Mr. *Huygens* obſerved long ago, that no voice or perfect inſtrument can always proceed by perfect intervals, without erring from the pitch at firſt aſſumed (*l*). But as this would offend the ear of the muſician, he naturally avoids it by his memory of the pitch, and by tempering the

(*l*) Aio itaque, ſi quis canat deinceps ſonos, quos Muſici notant literis C, F, D, G, C, per intervalla conſona, omninò perfecta, alternis voce aſcendens deſcendenſque; jam poſteriorem hunc ſonum C, toto commate, quod vocant, inferiorem fore C priore, unde cani cœpit. Quia nempe ex rationibus intervallorum iſtorum perfectis, quæ ſunt 4 ad 3, 5 ad 6, 4 ad 3, 2 ad 3, componitur ratio 160 ad 162, hoc eſt 80 ad 81, quæ eſt commatis. Ut proinde, ſi novies idem cantus repetatur, jam propemodum tono majore, cujus ratio 8 ad 9, deſcendiſſe vocem, tonoque excidiſſe oportet. *Coſmotheoros* lib. 1. pag. 77.

the intervals of the intermediate ſounds, ſo as to return to it again (*m*).

This is alſo confirmed by what we are told of a monk (*n*), who found, by ſubtracting all the aſcents of the voice in a certain chant from all its deſcents, that the latter exceeded the former by two commas: ſo that if the aſcents and deſcents were conſtantly made by perfect intervals, and the chant were repeated but four or five times, the final ſound, which in that chant ſhould be the ſame as the initial, would fall about a whole tone below it. But finding that the voices in his choir did not vary from the pitch aſſumed, he concluded that the muſical ratios, whereby he meaſured thoſe ſucceſſive aſcents and deſcents, were erroneous. But if he had known Mr. *Huygens*'s remark, it would have ſolved his difficulty.

This was not the firſt time that the truth of thoſe muſical ratios had been called in queſtion. For *Galileo* obſerved that the reaſon commonly al-

(*m*) Hoc verò nequaquam patitur aurium ſenſus, ſed toni ab initio ſumpti meminit, eodemque revertitur. Itaque cogimur occulto quodam temperamento uti, intervallaque illa canere imperfecta; ex quo multo minor oritur offenſio. Atque hujuſmodi moderamine ferè ubique cantus indiget; uti colligendis rationibus, quemadmodum hic fecimus, facile cognoſcitur. ibid.

(*n*) Methode generale pour former les Syſtème temperés de muſique. Mem. de l'Acad. des Scienc. Ann. 1707. pag. 263. 8vo.

alledged for it, appeared to him insufficient (*o*). At last indeed he hit upon a couple of experiments which gave him satisfaction (*p*), but a scientific proof was still wanting till Dr. *Taylor* published his theory of the vibrating motion of a musical chord (*q*), which has since been cultivated by several able mathematicians (*r*), and being the principal foundation of Harmonics, deserves to be further considered in the next section.

SECTION XI.

Of the vibrating motion of a musical chord.

PROPOSITION XXIII.

When a musical chord vibrates freely, the force which urges any small arch of it towards the center of its curvature,

(*o*) Stetti lungo tempo perplesso intorno à queste forme delle consonanze, non mi parendo che la ragione, che communemente se n' adduce da gli autori, che sin quì hanno scritto dottamente della musica, fosse concludente à bastanza. Dicono essi &c. *Discorsi attenenti alla Mecanica, Dialogo* 1°, towards the end.

(*p*) Ibidem.

(*q*) Methodus incrementorum, prop. 22, 23, and Philos. Trans. N° 337, IV. or Abrigd. by Jones Vol. 4. p. 391.

(*r*) Commentarii Acad. Petropol. Tom. III.
Comment on the Principia Vol. 2. pag. 347.
Mr. Maclaurin's Fluxions, art. 929.

ture, is to the tenſion of the chord in the ultimate ratio of the length of that arch when infinitely diminiſhed, to the radius of its curvature.

I ſuppoſe the chord to be uniform, and very ſlender, or rather to be a mathematical line, flexible by the leaſt force and elaſtic; and its tenſion or quantity of elaſtic force to be meaſured by a weight, which if hung to one end of it, would diſtend it to the ſame length which it has when it vibrates freely by the force of its elaſticity.

Pl.xxi. Fig. 50. Let ACB repreſent ſuch a chord fixed at the points A and B, CD any arch of it, CE and DE tangents at C and D; in either of which as EC produced if need be, take EF equal to ED, and draw FG perpendicular to FE, and DG to DE, and joining DF produce GE towards H.

Then imagine the chord to keep its curvature while a force applied at E or H draws the tangents EC, ED and theſe the points of contact C, D, ſo as to keep them in *æquilibrio*. And ſince the elaſtic forces at C and D are each equal to the force of tenſion, the direction of the third force at E will biſect the angle CED under the other two directions, and conſequently will coincide with the line GEH, agreeably to the conſtruction of the equal triangles EDG, EFG.

Hence the three forces at *H*, *C* and *D*, which would keep the point *E* at reſt, are proportional to the ſides *DF*, *FG*, *GD* of the iſoſcelar triangle *DFG*, to which their ſeveral directions *EH*, *EC*, *ED* are perpendicular; becauſe that triangle is ſimilar to any other, as *EDI*, whoſe ſides are either parallel to, or in part coincident with thoſe directions, and therefore proportional to the forces acting in them, by the known theorem in Mechanics (*s*).

Now ſuppoſe *CL* and *DM* to be the *radii* of the curvatures of the chord at the points *C*, *D*, and the curve *LM* to be the *locus* of all the centers of the curvatures at every point of the arch *CD*. Then conceiving the point *D* to move up to *C*, and conſequently *M* and *G* up to *L*, the limit of the variable ratio *DF* to *FG*, of the ſaid forces, will be that of the evaneſcent arch *CD* to *CL* the *radius* of its curvature. And a force conſtantly equal and oppoſite to the former of the two, is that which urges the vaniſhing arch *CD* in the direction *EG*, which ultimately coincides with *CL*; and the latter was the force of tenſion. Q. E. D.

Coroll. When a muſical chord vibrates freely, the forces which accelerate its ſmalleſt equal arches, are conſtantly proportional to their curvatures very nearly, provided the latitude of the vibrations be very ſmall in proportion to the length of the chord.

For

(*s*) See Theorem 33 of Keill's Phyſics.

For the force of its tenſion being then very nearly invariable, the forces which accelerate its ſmalleſt equal arches are very nearly in the inverſe ratio of the *radii* of their curvatures (*t*), which is the ſame as the direct ratio of the curvatures themſelves.

DEFINITION OF THE HARMONICAL CURVE.

Fig. 51. *Let C be the common center of any two circles DF, EG, and CDE, CFG any two ſemidiameters, and of either of the included arches as DF, let FH be the ſine, in which produced both ways, let the lines HI and HK be ſeverally equal to the other arch EG; then while the ſemidiameter CFG moves round the center C and carries with it the line IFHK, parallel to it ſelf and conſtantly equal to twice the arch EG, the extremities I, K will deſcribe a curve whoſe vertex is D and axis DC, and whoſe baſe ACB is equal to the ſemicircumference of the circle EG.*

(*t*) By the preſent Propoſition.

Coroll.

Coroll. 1. Pl. XXII. Fig. 52. Drawing FL perpendicular to the base ACL, a line KP perpendicular to the curve at K, will be parallel to EL.

For drawing KN perpendicular to the base, let the radius CFG go forwards a little into the place Cfg, and carry the line $KHFI$ into the place $khfi$, cutting KN in O and FL in r. Then since $HK = EG$ by the definition, and also $hk = Eg$, their difference Ok is $= Gg$. Now by the similar triangles CLF and Frf, CFf and CGg, OKk and NPK, we have $CL : CF :: Fr : Ff$, and $CF : CG :: Ff : Gg$, and *ex æquo* $CL : CG$ or $CE ::$ ($Fr : Gg :: OK : Ok ::$) $NP : NK$. Consequently the right angled Triangles CLE, NPK are equiangular, and the perpendicular KPM is parallel to the line EL.

Coroll. 2. At any point K the radius of curvature $KM : LE :: LE$ *quad.* $: KN \times CE$.

For drawing fl parallel to FL; another line kM, perpendicular to the curve at k, will be parallel to El, by coroll. 1; consequently if the arch Kk be infinitely diminished, either of the coinciding perpendiculars KM, kM will be the radius of the curvature at K.

In the line El take $Es = EL$ and joining Ls, the triangles LEs and KMk, CEL or CEl and sLl, are ultimately equiangular.

Now KN or $FL : LC :: fr : rF$ or KO, and $LC : LE :: PN : PK :: KO : Kk$, and *ex æquo*, $KN : LE :: fr : Kk$;

But

But $CE : LE :: sL : Ll$ or fr, therefore *componendo*, $KN \times CE : LE$ *quad.* $:: (sL : Kk ::)$ $LE : KM$.

Coroll. 3. Hence if the ratio of the circles CEG, CDF be vaſtly great, the curvature at any point K will be extremely ſmall, and its radius $KM : CE :: CE : KN$ very nearly; becauſe the lines LE and CE will be very nearly equal.

Coroll. 4. Upon the ſame ſuppoſition, the very ſmall curvatures at any points D, K are very nearly in the ratio of their diſtances DC, KN from the baſe AB.

For when CE and conſequently AB is given, the curvature at K, being reciprocally as its radius KM, is directly as KN by coroll. 3.

Coroll. 5. Fig. 53. While the greater circle remains let the leſſer be diminiſhed, and the curve $AKDB$ will be changed into another $A\varkappa\delta B$ of the ſame *ſpecies*, and every ordinate to the common baſe will be diminiſhed in the ſame ratio, that is, $NK : N\varkappa :: CD : C\delta$.

Fig. 52. For while any arch EG equal to HK or CN is given in magnitude, let the other radius CD or CF be diminiſhed, and becauſe the triangle CFH retains its *ſpecies*, the line CH or NK is diminiſhed in the ſame ratio with CF or CD.

Coroll. 6. Fig. 53. When the axes CD, $C\delta$ of two curves are very ſmall in compariſon to their common baſe AB, the curvatures at the tops of any

any two coincident ordinates NK, $N\varkappa$, are in the ratio of the ordinates.

For if $\varkappa\mu$ be the radius of curvature at $\varkappa$, by coroll. 3 we have $KM \times KN = CE$ *quad.* $= \varkappa\mu \times \varkappa N$; whence $\varkappa N : KN :: KM : \varkappa\mu$, that is, as the curvature at $\varkappa$ to the curvature at K.

Coroll. 7. Hence, ſuppoſing the curve $AKDB$ to have the elaſticity and tenſion of a muſical chord, it will vibrate to and fro in curves very nearly of the ſame ſpecies with the given curve $AKDB$, provided none of the vibrations be too large.

For let the firſt effort of the tenſion reduce that curve into ſome other, as $A\varkappa\delta B$, in the firſt moment of time; and ſince the ordinates DC, KN are in proportion as the curvatures at D and K by coroll. 4, and thoſe curvatures as the accelerating force at D and K (*u*), acting in the directions DC, KM or KN very nearly, and theſe forces as the velocities generated by them in that time, and the velocities as the naſcent ſpaces $D\delta$, $K\varkappa$; *alternando*, we have $DC : D\delta :: KN : K\varkappa$ and *dividendo*, $DC : \delta C :: KN : \varkappa N$. Conſequently by coroll. 5 the curve $A\varkappa\delta B$ is very nearly of the ſame ſpecies with $AKDB$. And in the next moment it will be changed into another of the ſame ſpecies, and ſo on, till every point of the chord be reduced to the baſe AB at the ſame inſtant. And by the motion here acquired it will be carried towards the oppoſite ſide of the baſe, till by the oppoſition of the tenſion, it ſhall loſe all

(*u*) Coroll. prop. XXIII.

all its motion by the ſame degrees, and in the ſame curves, by which it was acquired; and thus the chord will continually vibrate in curves of the ſame ſpecies as the firſt, neglecting the ſmall difference in the directions *KN*, *KM*, and the reſiſtance of the air.

Coroll. 8. The ſmall vibrations of a given muſical chord are iſochronous.

For if the chord at the limit of its vibration aſſumes the form of the harmonical curve, it will vibrate to and fro in curves of that ſpecies by coroll. 7, and its ſeveral particles, being accelerated by forces conſtantly proportional to their diſtances from the baſe *AB* (*x*), will deſcribe thoſe unequal diſtances in equal times, like a pendulum moving in a cycloid.

If the chord at the limit of its vibration aſſumes any other form, it will cut an harmonical curve, equal in length to it, in one or more points, as *A*, *K*, *L*, *B* in Fig. 54; and the intercepted parts of the chord will be more or leſs incurvated towards *AB* than the correſponding parts of the curve, according as they fall without or within them; and will accordingly be accelerated by greater or ſmaller forces than thoſe of the correſponding parts of the curve (*y*). Therefore, ſuppoſing the chord and curve to differ in nothing but their curvatures, the difference of the curvatures of the correſponding parts will be continually diminiſhed by the difference of their

(*x*) Cor. prop. XXIII and cor. 6. Defin. curve.
(*y*) Cor. prop. XXIII.

their forces, till the parts coincide either before, or when they arrive at the baſe *AB*. And thus the times of the ſeveral vibrations of the chord will be the ſame as thoſe of the curve, and therefore equal to one another.

Coroll. 9. The Figure contained under the harmonical curve and its baſe, is of the ſame ſpecies as the Figure of Sines.

Fig. 52. For ſuppoſing the circle *DFQ* to grow bigger till it becomes equal to *EGR*, the figure *AKDB* will become a figure of ſines. Becauſe any ordinate *KN* to the abſciſs *AN* or arch *GR*, being conſtantly equal to *FL*, will then be equal to the ſine of the arch *GR*; and thus every ordinate as *KN* is increaſed in the given ratio of *CF* to *CG*, or *CD* to *CE*. And on the contrary the ſeveral ordinates in the ſaid figure of ſines diminiſhed in that conſtant ratio of *CE* to *CD*, are the ordinates in the figure *AKDB* of the harmonical curve.

PROPSITION XXIV.

The vibrations of a muſical chord ſtretched by a weight, are iſochronous to thoſe of a pendulum, whoſe length is to the length of the chord, in a compound ratio of the weight of the chord to the weight that ſtretches it, and of the duplicate ratio of the diameter of a circle to its circumference.

Fig.

Pl. XXIII. Fig. 54. If P be the weight that ſtretches the chord ADB, and DCM be the radius of its curvature at the vertex D, the force that urges any ſmall particle Dd towards C is $= \frac{Dd}{DM} \times$ P by prop. XXIII.

And ſince Dd vibrates like a pendulum (z), if it were ſuſpended by a ſtring $OP = DC$ in a cycloid $QPR = 2DC$ or DCF, and were urged at the higheſt points Q, R by a force acting downwards like that of gravity, but equal to the ſaid force $\frac{Dd}{DM} \times$ P, which urges Dd at the limits D, F of its vibrations; the times of thoſe oſcillations and of theſe vibrations would be equal to one another. Becauſe the forces being alſo equal at all other equal diſtances of the particle from P and from C, would impel it through equal parts of the equal lines QP, DC in equal times.

Again putting p for the weight of the chord ADB or ACB, the weight of its particle Dd is $= \frac{Dd}{AB} \times p$.

Hence if another ſtring L be to the ſtring OP or DC, as this latter weight $\frac{Dd}{AB} \times p$ is to the former $\frac{Dd}{DM} \times$ P, equivalent to the force at D; and the particle Dd be again ſuſpended by the ſtring L in another cycloid of the length 2 L; ſince

(z) Coroll. 8. Def. of the curve.

ſince at the higheſt points of this cycloid the particle is urged downwards by the whole force $\frac{Dd}{AB} \times p$ of its own gravity, its oſcillations will be iſochronous to thoſe of the former pendulum (*a*). Becauſe we took their lengths in the ratio of the forces that act upon them at the higheſt points of the cycloids, that is, $L : DC :: p \times DM : P \times AB$; which two ratios compounded with DC to AB, give $L : AB :: p \times DM \times DC$ or $p \times CEq$ (*b*) $: P \times ABq$, which was to be proved. For CEq is to ABq in the duplicate ratio of the diameter to the circumference, by the definition of the curve; and we ſhewed above that every particle of the curve vibrates in the ſame time with the middlemoſt. Q. E. D.

Coroll. 1. The time of one ſemivibration, forwards or backwards, of the chord AB meaſured by inches and decimals, is $\frac{113}{355} \sqrt{\frac{p}{P} \times \frac{AB}{39.126}}$ and its reciprocal is the number of ſuch vibrations made in one ſecond.

For the length of a pendulum that vibrates forwards or backwards in one ſecond, is 39.126 inches in the latitude of London, and the diameter is to the circumference of a circle as 113 to 355 very nearly, and the times of the vibrations of pendulums are in the ſubduplicate ratio of their lengths. Whence putting t for that of the

(*a*) Coroll. Theor. 4 De Motu Pend. in Mr. *Cotes*'s Harmon. Menſurarum.

(*b*) Coroll. 3. Defin.

the pendulum L, we have $t'' : 1'' :: \sqrt{L} = \frac{113}{355}\sqrt{\frac{p}{P} \times AB} : \sqrt{39.126}$, and $t'' = \frac{113}{355} \sqrt{\frac{p}{P} \times \frac{AB}{39.126}}$, and $\frac{1}{t} = \frac{355}{113}\sqrt{\frac{P}{p} \times \frac{39.126}{AB}}$, the number of ſemivibrations made in one ſecond.

Coroll. 2. Suppoſing the laſt number to be n, we have the logarithm of $n^2 = \log. \frac{P}{p \times AB} +$ 2, 58676. 52698, which gives n very expeditiouſly.

For the logarithm of $\overline{\frac{355}{113}}\Big|^2 \times 39.126 =$ 2, 58676. 52698.

Coroll. 3. If the lengths and tenſions of two chords be equal, the times of their ſingle vibrations are in the ſubduplicate ratio of their weights, by coroll. 1.

Coroll. 4. If their lengths and weights be equal, the times of their ſingle vibrations are reciprocally in the ſubduplicate ratio of their tenſions, by coroll. 1.

Coroll. 5. If their tenſions be in the ratio of their weights, the times of their ſingle vibrations are in the ſubduplicate ratio of their lengths, by coroll. 1.

Coroll. 6. The weights of cylindrical chords are in a compound ratio of their ſpecific gravities, lengths and ſquares of their diameters, that is, p is as $s \times AB \times d^2$; whence t is as $AB \times d \sqrt{\frac{s}{P}}$, by coroll. 1.

Coroll. 7. Hence, if the tensions and diameters of homogeneal chords be equal, the times of their single vibrations are in the ratio of their lengths.

Coroll. 8. If the tensions and lengths of homogeneal chords be equal, the times of their single vibrations are in the ratio of their diameters.

Coroll. 9. If the tensions of similar chords be as their specific gravities, the times of their single vibrations are in the duplicate ratio of their lengths or of their diameters (*c*).

Scholium.

1. Hence we may find the number of vibrations made in a given time by any musical sound, by comparing it with the sound of a given chord stretched by a given weight.

For example in the experiment abovementioned (*d*) I found the length of the vibrating chord $AB = 35.55$ inches and its weight $p = 31$ grains troy: And the sound of it, when stretched by the weight $P = 7$ pounds averdupois $=$ 49000 grains troy, was two octaves below the sound of the pipe *d* there mentioned. Hence by coroll. 2, we have $n = 131.04$, the number of semivibrations made in one second by the wire *AB*,

(*c*) See *Galileo*'s experiménts on chords. Dialogo 1° attenente alla Mecanica, towards the end.

(*d*) Prop. XVIII.

AB, and $4n = 524.16$, the number of ſemivibrations made by $\frac{1}{4}AB$, by coroll. 7, or by the pipe d; which is double the number 262 of its whole vibrations.

Before this experiment was made the orifice of the pipe was cut perfectly circular, and then the length of the cylindrical part was exactly 21.6 inches, and its diameter 1.9, which I mention becauſe the experiment, being accurately made, is of uſe upon other occaſions.

2. When the thermometer is at Temperate, the latitude of a pulſe of the ſound of that pipe is to the length of the pipe, almoſt as $2\frac{1}{2}$ to 1, by prop. L. Lib. II. Princip. Philoſ.

ADVERTISEMENT.

THOUGH the theory of imperfect consonances has been demonstrated pretty clearly, I think, in the sixth Section, yet as I had considered some parts of it in different lights and searched a little further into some others for my own diversion, I thought it not amiss to print my papers in the form of the following Additions; that if the reader should desire any further information, he may have recourse to them whenever he pleases.

THE CONTENTS.

Schol. 5 to prop. xx, ſhewing the methods of altering the pitch of an organ pipe in order to tune it.

Scholium to Prop. viii.

Fig. 63. WHEN different multiples as 3 *AB* and 2 *ab* of the vibrations *A B*, *ab* of imperfect uniſons, are the ſingle vibrations *AD*, *ac* of an imperfect conſonance, the multipliers 3 and 2 are in the ratio of the ſingle vibrations 3 *AB* and 2 *AB*, or 3 *ab* and 2 *ab* of the perfect conſonance, and therefore ſhould be irreducible to ſmaller numbers. The different multiples of the vibrations of imperfect uniſons are therefore ſuppoſed in the propoſition to be the leaſt in the ſame ratio.

Pl. xxv. Fig. 63, 64. But if different multiples of the vibrations *AB*, *ab*, as 6 *AB* and 4 *ab*, whoſe multipliers 6 and 4 are reducible by a common diviſor, be the ſingle vibrations of an imperfect conſonance, (as they may by intermitting 6—1 pulſes of *AB* and 4—1 of *ab*, ſo as to leave ſingle pulſes at firſt and between every intermiſſion,) the period of the imperfections of this conſonance will not be equal to that of the imperfect uniſons *AB*, *ab*, but multiple of it by 2, the greateſt common diviſor of the multipliers 6 and 4.

For thoſe multiple vibrations 6 *AB* and 4 *ab* are the ſame as 3×2 *AB* and 2×2 *ab*, or 3 *AC* and 2 *ac*, in which the equimultiples 2 *AB* and 2 *ab*, or *AC* and *ac* in fig. 64, are the ſingle vibrations of other imperfect uniſons, reſulting from an intermiſſion of every ſecond pulſe of *AB* and *ab* in fig. 63; and the period of their imperfections is equal to that of 3*AC* and 2*ac*, or *AG* and *ae* by this VIII^th^ propoſition, and is the ſame multiple of the period of *AB* and *ab*, as *AC* is of *AB*, or *ac* of *ab* by the coroll. 4 to this propoſition, that is by 2, the greateſt common diviſor of the multipliers 6 and 4.

An illuſtration of Prop. x.

THUS in Fig. 23, Pl. XI, after taking away 9 ſhort cycles from each end of the cycle *AU* of imperfect uniſons, there remains *kKLm*, part of two more; and in Fig. 25, after taking away 3 ſhort cycles from each end of the period *AX*, there remains *dDe*, part of another; and in Fig. 27, after taking away 4 ſhort cycles of imperfect octaves from each end of the period *Aw*, there remains *iILm*, part of 2 more; and laſtly in Fig. 34, Pl. XII, after taking away 2 ſhort cycles of imperfect V^ths^ from each end of the cycle *AZ* of the imperfect uniſons, there remains *nNQr*, part of another: which though not ſituated exactly in the middle of *AZ*, by reaſon of the part $\Delta\varepsilon Z$ of another ſhort cycle, containing equimultiples of *AB* and *ab*, is comparatively very near it when the number of ſhort cycles in the period is very large as uſual; in which caſe the beats will be made very nearly in the middle of every period.

Scholium.

Scholium.

It is very unreaſonable to ſuppoſe with Mr. *Sauveur* that the beats are made by the united force of the coincident pulſes of imperfect uniſons (*e*).

For while the imperfect uniſons are made to approach gradually to perfection, experience ſhews that they always beat ſlower and ſlower (*f*) and by theory (*g*) the periods of their pulſes grow longer and longer. Therefore in conſequence of that gentleman's hypotheſis, the uniſons ſhould alſo beat at the ends of the periods where the pulſes do not coincide: Becauſe it is very improbable that the cycle of uniſons, ſuppoſing it ſimple at firſt, while it lengthens gradually, will not ſometimes be changed into periods as well as into other ſimple cycles.

Nor can it be allowed that the uniſons will beat only at the ends of their complex cycles. For according as the numeral terms expreſſing the ratios of the ſingle vibrations of the ſeveral ſucceſſive uniſons, happen to be reducible or not reducible or to be irrational, the cycles of the pulſes will ſometimes be ſhortened, ſometimes lengthened again, ſometimes invariable and ſometimes impoſſible, as ſhall be explained by and by; which accidents diſagree with the conſtant gradual retardation of the beats in the preſent caſe.

(*e*) Prop. XI. ſchol. 3.
(*f*) See Phænomena of beats placed before prop. X.
(*g*) Cor. 5. lemma to prop. IX.

If it be ſaid that the pulſes next to the periodical points fall ſo cloſe to one another, as to affect the ear in the ſame manner as if they were quite coincident; it may be ſo, and moſt probably is ſo. And then it will follow that the harmony of the ſhort cycles terminated by ſuch cloſe pulſes, will there be much the ſame as that of perfect uniſons; at leaſt it will certainly be better about the periodical points and coincident pulſes than any where elſe in the periods. But the ſound of a beat has no harmony in it; on the contrary it rather reſembles the common ſound of a beat or ſtroke upon any groſs, irregular body: And this ſound reſults from pulſes of air which rebounding from different parts of the body, diſpoſed to vibrate in different times, will ſtrike the ear one after another at irregular intervals, like the pulſes in the middle between the periodical points of imperfect uniſons. Therefore theſe are the only pulſes in each period, which can excite the ſenſation of the beats of imperfect uniſons. And the like argument is applicable to any other imperfect conſonance by prop. VIII.

Pl. XXIV. Fig. 55. As to thoſe uncertain lengths abovementioned of the ſimple and complex cycles of the pulſes of imperfect uniſons, while their interval is continually diminiſhed or increaſed; let one of the ſounds be fixt and the time of its ſingle vibration be repreſented by any given line *V* and thoſe of the variable ſound by the ſucceſſive lines *A*, *B*, *C*, &c, all which lines may conſtitute any increaſing or decreaſing progreſſion; and

and ſuppoſing n to repreſent any large given number, let $A : V :: n : a$, $B : V :: n : b$, $C : V :: n : c$, &c.

Then will the cycles of the pulſes of V and A, V and B, V and C, &c, be $nV = aA$, $nV = bB$, $nV = cC$, &c, provided every one of the numbers a, b, c, &c, be integers and primes with reſpect to the aſſumed number n. In which caſe the ſeveral cycles are equal to one another and to nV.

But if the terms of all or any of thoſe ratios have a common diviſor, the correſponding cycles will be ſhortened in proportion as the greateſt common diviſors are larger; and therefore their lengths cannot increaſe or decreaſe ſucceſſively in regular order while the ſucceſſive intervals of the uniſons continually decreaſe or increaſe, unleſs the greateſt common diviſors decreaſe or increaſe in regular order too; which can happen but very rarely.

And when the terms of the ratios of any of the vibrations happen to be incommenſurable, a ſecond coincidence of their pulſes will be impoſſible: becauſe no multiple of one vibration can be equal to any multiple of the other.

But in all caſes whatever, the periods of the pulſes of V and A, V and B, V and C, &c, which are $\frac{nV}{n-a}$, $\frac{nV}{n-b}$, $\frac{nV}{n-c}$, &c (*b*), will decreaſe continually in the ſame proportions with the fractions

(*b*) Def. III. ſect. VI.

tions $\frac{n}{n-a}$, $\frac{n}{n-b}$, $\frac{n}{n-c}$ whoſe magnitudes can never be altered by any common diviſors of their terms, whether integers fractions or ſurds.

Another demonſtration of ſcholium 5. Prop. xi, and of Prop. vii.

THE breadth of the apparent Undulations of the lights and ſhades ſeen at a diſtance upon two rows of parallel objects, may be alſo found by the following conſtruction.

Pl. XXIV. Fig. 56. Let a plane paſſing through a diſtant eye at z, cut the axes of the parallel objects at right angles in the points a, b, c, &c, α, β, γ, &c, which are ſuppoſed equidiſtant in both the parallel lines abc, $\alpha\beta\gamma$. From any object in one of theſe lines to any ſucceſſive objects in the other, draw the lines αa, αb, αc, &c, and the lines zvV, zxX, zyY, &c, drawn parallel to them, will intercept the equal breadths of the apparent undulations.

Becauſe while the eye is gradually directed from the middle of any of the breadths VX, XY, &c, towards either of its extremities, the objects will appear cloſer together in couples, in proportion to their ſmaller diſtances from the next extremity, which was ſhewn in this ſcholium to be the cauſe of the undulations.

For the lines $d\delta$, $e\varepsilon$, $f\zeta$, $g\eta$, &c, being parallel to $a\alpha$, are parallel to Vv by conſtruction; and the lines $i\theta$, $k\iota$, $l\varkappa$, $m\lambda$, &c, being parallel to

$b\alpha$,

$b\alpha$, are parallel to Xx; and ſo on. Let lines drawn from z through the objects δ, ε, ζ, &c, of one row, cut the line of the other in D, E, F, &c. Then becauſe the rows are parallel, the ratio of $D\delta$ to δz, $E\varepsilon$ to εz, $F\zeta$ to ζz, &c, is the ſame as Vv to vz, or Xx to xz, &c (i). Whence alſo, becauſe of the parallels between the rows, we have

$$\left.\begin{array}{l} Dd : dV :: D\delta : \delta z \\ Ee : eV :: E\varepsilon : \varepsilon z \\ Ff : fV :: F\zeta : \zeta z \\ Gg : gV :: G\eta : \eta z \\ Hi : iX :: H\theta : \theta z \\ Ik : kX :: I\iota : \iota z \\ Kl : lX :: K\kappa : \kappa z \\ Lm : mX :: L\lambda : \lambda z \\ \text{\&c.} \qquad\quad \text{\&c.} \end{array}\right\} :: Vv : vz;$$

That is, all thoſe ratios are equal, and, alternately, the leſſer apparent intervals Dd, Ee, Ff, Gg, are proportional to their diſtance dV, eV, fV, gV, from the next extremity V of the breadth VX; and alſo Hi, Ik, Kl, Lm, proportional to iX, kX, lX, mX, their diſtances from the next extremity X of the ſame breadth VX. And the breadths VX, XY, &c, are equal, becauſe ab, bc, &c are ſo, and the triangles VzX and $a\alpha b$, XzY and $b\alpha c$, &c, are ſimilar by conſtruction. Q. E. D.

Coroll. 1. The projections DE, EF, &c, of the equal intervals $\delta\varepsilon$, $\varepsilon\zeta$, &c, are to theſe intervals in

(i) 2 VI. Euclid.

in the conſtant ratio of Dz to δz, or Ez to εz, or Vz to vz, and conſequently are equal to one another. Therefore ſuppoſing the lines DE and $\delta\varepsilon$ or de to repreſent the times of the ſingle vibrations of imperfect uniſons, the periods of the neareſt approaches of their pulſes D, E, &c. d, e, &c, are VX, XY, &c; And in going from their extremities V, X to the middle, the alternate leſſer intervals between the ſucceſſive pulſes, are proportional to their diſtances from the next extremity, as we ſhewed juſt now: which is another proof of prop. VII.

Coroll. 2. If the eye be moved in a line parallel to the rows, the breadths of the apparent undulations will be conſtantly the ſame, and if it be moved uniformly in any other right line, their breadths will vary uniformly, and be conſtantly proportional to the diſtance of the eye from the rows. Becauſe the triangles VzX, VzY, &c, are conſtantly ſimilar to $a\alpha b$, $a\alpha c$, &c. And this concluſion ſeems to agree with what I have tranſiently obſerved of theſe undulations.

But it is eaſy to collect from the conſtruction of the figure, and the different ratios of zV to zv expreſſed by numbers, that the intervals between the apparent conjunctions of the objects will increaſe and decreaſe very irregularly; and that no conjunctions can happen except when the eye arrives at certain points of its courſe, and none at all, mathematically ſpeaking, when its diſtances from the two rows, meaſured upon any right line, happen to be incommenſurable.

Which

Which conclufions being contrary to the continual appearance of the undulations to the eye in all places, and to the regular increafe or decreafe of their breadth, fhew, that their breadth is not equal to the interval between the apparent conjunctions, no more than the interval between the beats of imperfect unifons is equal to the interval between their coincident pulfes.

LEMMA.

In any period between the fucceffive beats of an imperfect confonance, any given number of fhort cycles next to one fide of the leaft diflocation of the pulfes, is more harmonious, and the fame number of them next to the other fide is lefs harmonious than the fame number of them next to either fide of the coincident pulfes: and thefe degrees of harmony diff.r more in thofe periods where the two leaft diflocations differ lefs, and moft of all in the periods where thefe diflocations are equal when poffible.

Pl. xxv. Fig. 59. Let *AB* and *ab* reprefent the times of the fingle vibrations of imperfect unifons, *A* and *a* their coincident pulfes, *B*, *C*, *D*, &c, *b*, *c*, *d*, &c, their fucceffive pulfes on each fide of *A*, *a*; *Rr* their leaft diflocation in any given period, and confequently the neareft to the periodical point *z*, which is here placed under *A*, for the convenience of feeing at one view,

view, the ſhort cycles next to both ſides of Rr and Aa.

Firſt I ſay, the ſhort cycles RS, ST, &c, which include z, are more harmonious, and RQ, QP, &c, leſs harmonious than AB, BC, &c, the numbers of them being the ſame: and that the degrees of their harmony differ more in the periods where the two leaſt diſlocations Rr, sS differ leſs, and moſt of all where $Rr = sS$, when poſſible (*k*).

For $bB = (AB - Ab = RS - rs =)\ Rr + sS$ (*l*):
And $cC = (AC - Ac = RT - rt =)\ Rr + tT$.
&c. &c.

Hence the ſucceſſive diſlocations sS, tT, &c, are reſpectively ſmaller than bB, cC, &c, by Rr, as appears alſo by their ſmaller diſtances from z (*m*). But on the other ſide of Rr, the diſlocations Qq, Pp, &c, are reſpectively greater than bB, cC, &c, by the ſame Rr, for the like reaſons.

Now the ſhort cycle RS which includes z, is more harmonious than AB next to A. For though the diſlocations $Rr + sS$ are $= bB$, yet thoſe parts of bB, as being ſmaller than bB, will give leſs offence to the ear than the whole: the whole may be perceived and give ſome offence even when one or both its parts are imperceptible. And for the ſame reaſons the ſhort cycle RS will be ſtill more harmonious than AB in other periods

(*k*) See prop. VII. coroll. 2.
(*l*) See prop. VII. coroll. 1.
(*m*) Prop. VII.

periods where Rr, sS are leſs unequal, and the moſt harmonious where they are equal when poſſible (*n*): their ſum being every where the ſame.

The next ſhort cycle ST is alſo more harmonious than BC; the diſlocations sS, tT being reſpectively ſmaller than bB, cC. Therefore the ſhort cycles RS, ST, taken together, are more harmonious than AB, BC taken together; and ſtill more harmonious in other periods where sS, tT are ſmaller, till sS be equal to Rr.

But on the other ſide of Rr and Aa, the ſhort cycle RQ is leſs harmonious than AB, the diſlocations Qq, Rr being larger than Bb and o. The next ſhort cycle QP is alſo leſs harmonious than BC; the diſlocations Pp, Qq being reſpectively larger than Cc, Bb. Therefore RQ, QP together are leſs harmonious than AB, BC together; and ſtill leſs harmonious in other periods where Rr, Qq, Pp are larger, till Rr be equal to sS. And the ſame is evident in any larger equal numbers of ſhort cycles throughout the period between the ſucceſſive beats.

Secondly, any imperfect uniſons will be changed into imperfect octaves whoſe ſingle vibrations are AC and Ab, or Ac and AB, by conceiving every ſecond pulſe of the ſeries A, B, C, &c, or a, b, c, &c, to be intermitted, which would depreſs one of the uniſons an octave lower.

Now if that intermiſſion ſhould take away the alternate pulſes S, U, &c, or s, u, &c, the ſhort cycles

(*n*) Prop. VII. coroll. 2.

cycles of the octaves, next to one ſide of Rr, will be RT, TW, &c, and on the other, RP, PN, &c: I ſay the former as including z are more, and the latter leſs harmonious than AC, CE, &c, the numbers of them being equal.

For we had $Rr + tT = cC$, conſequently the ſhort cycle RT is more harmonious than AC, for the ſame reaſon as in uniſons, and becauſe the intermediate diſlocations sS, bB are vaniſhed, one of their conſtituent pulſes in each being taken away. And RT is ſtill more harmonious than AC in other periods where Rr and tT are leſs unequal.

The next ſhort cycle TW is alſo more harmonious than CE, the diſlocations tT, wW being reſpectively ſmaller than cC, eE, as in uniſons; and is ſtill more harmonious in other periods where tT, wW are ſmaller, that is where tT and Rr are leſs unequal.

But on the other ſide of Rr and Aa, the ſhort cycle RP is leſs harmonious than AC, and PN than CE, the diſlocations Rr, Pp being reſpectively bigger than o and Cc; and Pp, Nn bigger than Cc, Ee, reſpectively: and is ſtill leſs harmonious in other periods where Rr, Pp, Nn are larger, that is where Rr, tT are leſs unequal.

Therefore the ſhort cycles RT, TW, &c are more, and RP, PN, &c are leſs harmonious than AC, CE, &c.

Likewiſe if that alternate intermiſſion ſhould take away the pulſes R, T, W, &c, or r, t, w, &c, then the leaſt diſlocation is sS, and the ſhort cycles

cycles SQ, QO, &c, as including z, will be more, and SU, UX, &c leſs harmonious than AC, CE, &c, for the very ſame reaſons as before.

Thirdly, any imperfect uniſons will be changed into imperfect v^ths^, whoſe vibrations are Ac and AD, (or AC and Ad) that is $2ab$ and $3AB$, by intermitting 2—1 pulſes of the ſeries a, b, c, d, &c, which depreſſes the acuter uniſon an VIII^th^ lower, and 3—1 pulſes of the ſeries A, B, C, D, &c leaving ſingle ones between, which depreſſes the graver uniſon a XII^th^ or VIII + V^th^ lower; and thus the interval of the new ſounds is an imperfect v^th^, as repreſented in the uppermoſt parallel in the figure.

Now in the period where thoſe intermiſſions leave the pulſes r, t, w, y, &c, R, U, Y, &c, (as in the 4^th^ parallel) the intermediate ones will be taken away, and then Rr being the leſſer of the two diſlocations in the ſhort cycle RY which includes z, is the leaſt of all in this period. And the ſhort cycles RY, &c, on this ſide of Rr, will be more harmonious than AG, &c (in the firſt parallel); and on the other ſide, the ſhort cycles RL, &c, will be leſs harmonious than AG, &c: For the ſame reaſons as above.

Likewiſe in the period where the pulſes q, s, u, x, &c, Q, T, X, &c are left (in the 5^th^ parallel), the intermediate ones will be abſent, and then Qq is the leaſt diſlocation in this period, and a greater difference than before will be found in the harmony of the ſhort cycles on each ſide of

Qq and *Aa*; the difference *Xx*—*Qq* being leſs than *Yy*—*Rr* in the former caſe.

Laſtly in the period where *p*, *r*, *t*, *w*, &c, *P*, *S*, *W*, &c, are left (in the loweſt parallel), the intermediate ones are intermitted, and then *Pp* is the leaſt diſlocation in this period, and a difference ſtill greater will be found in the harmony of the ſhort cycles on each ſide of *Pp* and *Aa*, for the like reaſon. And the greateſt difference will be found where theſe diſlocations are equal when poſſible; that is, when a periodical point *z* biſects a ſhort cycle of any conſonance, which conſiſts of any odd number of thoſe of the uniſons; and alſo when either of the coincident pulſes at the ends of the complex or ſimple cycles of the uniſons, biſects a ſhort cycle of any conſonance conſiſting of any even number of thoſe of the uniſons as in Fig. 35. Plate XII. The like proof is plainly applicable to the vibrations *AC*, *Ad*, or to thoſe of any other conſonance. Q. E. D.

Coroll. Hence any two imperfect conſonances will be as equally harmonious as they poſſibly can be, when the periods (between their ſucceſſive beats) which are biſected by their coincident pulſes, are made equally harmonious; theſe periods having a mean degree of harmony among thoſe of all the other periods in each conſonance.

All thoſe degrees of harmony occur in practical muſic, and whether ſenſibly different or

not,

not (*o*), muſt be uſed as if they were equal, and in theory we muſt take the medium among them.

As the proof of this concluſion has been pretty long, I avoided it in the Book by a paragraph in the demonſtration of prop. XIII, which may now be proved ſomewhat differently.

Another demonſtration of Prop. xiii. and its third corollary.

Pl. XXV. Fig. 60, 61. Let op and OP repreſent the times of the ſingle vibrations of imperfect uniſons; ab and AB thoſe of other imperfect uniſons; o and O, a and A their coincident pulſes; and if $ab = op$, the period of the pulſes of the former uniſons, will be to that of the latter, ultimately as bB to pP (*p*).

1. Taking bB to pP as 1 to 2, this is now the ratio of the lengths of the periods of the uniſons op and OP, ab and AB; and the latter is of the ſame length as the period of the leaſt imperfections of octaves, whoſe ſingle vibrations are ab and AC or $2AB$, by intermitting $2-1$ pulſes of the ſeries A, B, C, D, &c, by prop. VIII.

Now the ſhorter length of the ſhort cycles of the uniſons op, OP, is $op = ab$, and that of the ſhort cycles of the imperfect octaves is ac or $2ab$,

 and

(*o*) Prop. XI. ſchol. 4. art. 5. laſt paragr.

(*p*) Cor. 8. lem. to prop. IX, and prop. XI. ſchol. 1.

and the ratio of their lengths is 1 to 2, which being the ſame as that of the periods of the uniſons and octaves, ſhews that their ſhort cycles are equally numerous in them.

The longer length of the ſhort cycle of the octaves is AC or $2AB$, and the difference of the lengths is $2AB - 2ab = 2bB = cC$ the diſlocation of the pulſes at the end of the firſt ſhort cycle, and is equal to pP, becauſe we took $bB : pP :: 1 : 2$; therefore the ſeveral diſlocations eE, &c, qQ, &c, at the ends of the ſubſequent ſhort cycles of the octaves and uniſons, are equal reſpectively throughout their half periods, which are therefore equally harmonious:

Becauſe thoſe diſlocations are the cauſes that ſpoil the harmony, more or leſs according as they are greater or ſmaller; and cauſes conſtantly equal muſt have equal effects: And becauſe the harmony of theſe half periods is the medium among the degrees of harmony of all the reſt, by the coroll. to the lemma.

2. Fig. 60, 62. Again, taking bB to pP as 1 to 3, this is now the ratio of the periods of the imperfect uniſons op and OP, ab and AB; and the latter period is equal to that of imperfect XII^ths, whoſe vibrations will be ab and AD or $3AB$ by intermitting $3-1$ pulſes of the ſeries A, B, C, D, &c, ſo as to leave ſingle pulſes between every intermiſſion (q). And ſince $Pp = (3bB =) dD$, it appears that the ſeveral ſubſequent diſlocations qQ, &c, gG, &c, of the uniſons

(q) Prop. VIII.

ſons and XII^{ths} are equally numerous and equal reſpectively throughout the half periods on each ſide of oO and aA; which render the conſonances as equally harmonious as they poſſibly can be, for the reaſons above mentioned.

3. Fig. 60, 63. Laſtly, taking bB to pP as 1 to 2×3, this is now the ratio of the lengths of the periods of the diſlocations of the imperfect uniſons op and OP, ab and AB, for the reaſon above. And the latter period is of the ſame length as that of imperfect V^{ths}, whoſe ſingle vibrations $2ab$ and $3AB$ reſult from intermitting $2-1$ pulſes of the ſeries a, b, c, d, e, f, g, &c, and $3-1$ pulſes of the ſeries A, B, C, D, E, F, G, &c, ſo as to leave ſingle pulſes at the beginning, and between every intermiſſion, by prop. VIII.

Now the ſhorter length of the ſhort cycle of the uniſons op, OP is $op=ab$, and that of the ſhort cycle of the imperfect V^{ths} is $2 \times 3ab$ (becauſe $2ab : 3ab :: 2 : 3$) and the ratio of theſe lengths is 1 to 2×3, the ſame as that of the periods of the imperfect uniſons op, OP and the V^{ths}, whoſe ſhort cycles op and ag are therefore equally numerous in them.

The longer length of the imperfect ſhort cycle of the V^{ths} is $2 \times 3AB$ (becauſe $2AB : 3AB :: 2 : 3$) and the difference of the longer and ſhorter lengths is $2 \times 3AB - 2 \times 3ab = 2 \times 3 \times \overline{AB - Ab} = 2 \times 3bB = gG$, the diſlocation of the pulſes at the end of that ſhort cycle, and is equal to pP,

 becauſe

because we took $bB : pP :: 1 : 2 \times 3$. Therefore the several dislocations nN, &c, qQ, &c at the ends of all the subsequent short cycles of the v^ths and unisons, are respectively equal in magnitude and number too, throughout the half periods on each side of the coincident pulses aA, oO; which equalities make these consonances as equally harmonious as they possibly can be, for the reasons above.

4. Instead of the terms 2 and 3 of the ratio of the vibrations of perfect v^ths, if we substitute those of any other perfect consonance, or m and n indeterminately for them, the method of demonstration will be evidently the same as in the last example.

Now those imperfect consonances of VIII^ths, XII^ths, v^ths, &c are not only equally harmonious with the same imperfect unisons op, OP, but also with one another; the dislocations pP, cC, dD, gG, at the ends of their first and subsequent short cycles, being equal and equally numerous in their periods. And since any one of them is equally harmonious to another of the same name at any other pitch, when their short cycles are equally numerous in their periods (r), it appears that all sorts of imperfect consonances are as equally harmonious as possible, when their short cycles are equally numerous in the periods of their imperfections. Q. E. D.

The equal harmony of flat consonances is demonstrable in the same manner.

Coroll.

(r) Prop. XII.

Coroll. Hence when imperfect consonances are equally harmonious, their temperaments have very nearly the inverse ratio of the products of the terms expressing the ratios of the single vibrations of the perfect consonances.

This is the third corollary to prop. XIII and may be demonstrated in this other manner.

The interval of the sounds of imperfect unisons is the temperament of the interval of any consonance whose single vibrations are different multiples of the vibrations of those unisons (*s*).

Now in all the examples of tempered consonances we made the vibration ab constant and AB variable. Consequently the several intervals of these imperfect unisons, or the logarithms of the ratios of ab to AB were very nearly proportional to the differences bB (*t*), which in the VIIIths and Vths were made equal to $\frac{1}{1 \times 2}\, pP$ and $\frac{1}{2 \times 3}\, pP$ respectively. Therefore when these consonances are equally harmonious, the ratio of their temperaments is $\frac{1}{1 \times 2}$ to $\frac{1}{2 \times 3}$ very nearly.

And when either of them is equally harmonious to another of the same name at a different pitch, their temperaments are equal (*u*), and the terms of the ratio of the vibrations of the

(*s*) Prop. VIII. cor. I.
(*t*) Cor. I. lem. to prop. IX.
(*u*) Prop. XII. coroll.

the perfect consonances of that name are the same.

Consequently the direct ratio of the temperaments and the inverse ratio of the products of those terms, are very nearly the same in all equally harmonious consonances.

An illustration of coroll. 1. *Prop. xiii.*

Pl. XXIV. Fig. 57. Let the intervals of the equidistant points A, I, II, III, &c be the longer or the shorter lengths of the imperfect short cycles of any given consonance; whose half period is AP; A and a its coincident pulses; ab the lesser of the vibrations of the imperfect unisons whose half period is also AP. Make the perpendicular $PQ = \frac{1}{2} ab$, and draw AQ cutting the perpendiculars at I, II, III, &c, in D, D, D, &c. Then are these perpendiculars equal to the dislocations of the pulses between the successive short cycles of the imperfect consonance, by prop. VII and VIII.

Fig. 58. Make the like construction denoted by the greek letters for any other imperfect consonance of the same or a different name. And if it be equally harmonious to the former, its half period $\alpha\pi$ will contain the same number of short cycles as AP does (x); suppose 6 in each. By lessening its temperament, let its half period be lengthened to αp, where erecting the perpendicular

(x) Prop. XII, XIII.

dicular $pq = \pi\varkappa$ join αq cutting all the intermediate perpendiculars in e, e, &c. Then the several new dislocations $1e$, $2e$, $3e$, &c will be smaller than 1δ, 2δ, 3δ &c respectively. Therefore the short cycles $\alpha 6 e$, contained in a part of the new half period αp, are not only more harmonious than the short cycles $\alpha 6 \delta$, contained in the old half period $\alpha\pi\varkappa$, or than $AVID$, but those in the remaining part $e678e$ continue the harmony in the new half period αp, after that of the old half period is quite extinguished by the beat at the end of it.

Coroll. Since only the corresponding short cycles of imperfect consonances can admit of a just comparison, one by one, in the order of their succession, beginning from the coincident pulses, or from their least dislocations next to the periodical points, (as explained in the demonstration of the lemma,) if the periods of two consonances contain unequal numbers of their short cycles, the comparison will be imperfect; which is another argument *à priori* for the truth of the XII[th] and XIII[th] propositions.

Scholium I.

In any pure consonance (y) the short cycle contains but one vibration of the base, as in Fig. 61, 62, and the equal times between the pulses of the treble are never subdivided by any pulses of the base, except at the ends of the short

(y) Sect. III. art. 8.

ſhort cycles; and here the diſlocations *cC*, *dD* are conſidered and adjuſted with the analogous ones in other pure conſonances, by the XIIIth propoſition.

But in any other conſonance whoſe ſhort cycle contains ſeveral vibrations of the baſe, the equal times between the pulſes of the treble are ſubdivided by the pulſes of the baſe, not only at the ends of the ſhort cycles, but between them, as at *D*, *K*, &c, Fig. 63; where the conſideration of the inequalities of the intervals *cD* and *De*, *iK* and *Kl*, &c, ſeems to have been neglected in the ſaid propoſition, but in reality is implied in it.

Fig. 63. For ſuppoſing the alternate pulſes *D*, *K*, &c to be intermitted or taken away, thoſe V^{ths} will be changed into XII^{ths} an octave lower than the XII^{ths} in Fig. 62; but will not be equally harmonious with them, as the V^{ths} were ſuppoſed to be before that intermiſſion, till the diſlocations, *gG*, *nN*, &c, in Fig. 63 be doubled; that the temperaments of both the XII^{ths} may be equal and their periods proportional to their vibrations and ſhort cycles (*z*).

While the diſlocations *gG*, *nN*, &c, remain doubled, reſtore the pulſes *D*, *K*, &c, to their places, and now the intermediate inequalities *Dc*—*De*, *Ki*—*Kl*, &c are alſo the doubles of their former magnitudes and the new V^{ths} are leſs harmonious than the XII^{ths} in Fig. 62, and will not be equally harmonious with them till the diſlocation,

(*z*) Prop. XII. coroll.

diſlocation, *gG*, *nN*, &c, and conſequently the inequalities *Dc*—*De*, *Ki*—*Kl*, &c be contracted to their former magnitudes.

Therefore theſe interrupted conſonances are not conſidered as pure ones in the XII^th^ and XIII^th^ propoſitions, but allowance is made on courſe for the effect of the intermediate pulſes of the baſe.

Scholium 2.

Pl. XXV. Fig. 63. Suppoſing the letters *d*, *k* to be reſtored to the places of the abſent pulſes of the imperfect uniſons, that fall next to *D* and *K*, I call the lines or times *dD*, *kK* the Aberrations of the interior pulſes *D*, *K*, from the places *d*, *k* which they have in the perfect ſhort cycles. Likewiſe in the upper part of the ſame figure, if *AE* and *ad* be the ſingle vibrations of an imperfect 4^th^, then *Ee* is an aberration of one of the interior pulſes of the baſe in the firſt ſhort cycle.

Now if the ratio of the times of the ſingle vibrations of any perfect conſonance be *m* to *n* in the leaſt integers, and when it is tempered, if $2D$ be the ſum of the exterior diſlocations in any given ſhort cycle, the aberration or ſum of the aberrations of the interior pulſes of the baſe, from the places they have when the conſonance is perfect, will be $\overline{n-1} \times D$.

The reaſon of the theorem will ſoon appear by drawing a ſhort cycle or two of a 4^th^, III^d^, &c,

&c, and by obſerving, that as n is the number of the vibrations of the baſe contained in any ſhort cycle, ſo $n-1$ is the number of its pulſes excluſive of the extreams, and that the ſum of the exterior diſlocations is equal to the ſum of any two interior aberrations equidiſtant from them, or to double the aberration in the middle; as is plain from the arithmetical progreſſion of the alternate leſſer intervals of the imperfect uniſons, from which the given conſonance is derived.

Therefore in two equally harmonious conſonances, as the ſum of the exterior diſlocations in any ſhort cycle of the one, is to the ſum of them in the correſponding ſhort cycle of the other, in a certain conſtant ratio (*a*), ſo the interior aberration or the ſum of the interior aberrations in the former ſhort cycle, is to the ſum of them in the latter in another conſtant ratio; and *componendo*, the totals of the exterior diſlocations and interior aberrations are alſo in another conſtant ratio.

But the temperaments and periods of the two conſonances muſt be adjuſted by the firſt given ratio alone, without any regard to the ſecond or third.

1. Becauſe the exterior diſlocations are of a different kind from the interior aberrations. For as in ſeeing ſo in hearing, it is more difficult for the ſenſe to perceive the quantity of a ſmall inequality in the larger ſucceſſive interval of the points or

(*a*) By the foregoing illuſtration.

or pulses *c*, *D*, *e*, Fig. 63, than to perceive the same or a different small quantity when bounded by two visible points or audible pulses *g*, *G*. And the difficulty is greater in more complex short cycles of imperfect 4ths, IIIds, VIths, &c, where the successive intervals between the points analogus to *c*, *D*, *e*, do not err from the simplest ratio of 1 to 1, but from the more complex ones of 1 to 2, 1 to 3, &c; as will easily appear from the disposition of the pulses in such cycles in Fig. 5, Plate 1. For which reason the ratio of the sum of the interior aberrations ought not to be compared and compounded with that of the exterior dislocations.

2. Because it appears from the corollary to the foregoing Illustration, that no other regard can be had to the interior aberrations, than what follows on course from the given ratio of the exterior dislocations, determined by the equality of their numbers in the periods of the two consonances, as in prop. XII and XIII.

Scholium 3.

To give the reader more determinate ideas of the numbers of vibrations, short cycles and dislocations contained in the long cycles and periods of imperfect consonances, and of their absolute duration in practical music, I will add a computation of them in a consonance of Vths tempered

pered by $\frac{1}{4}$ comma, as it ufually is, more or lefs, in organs and harpfichords.

Pl. XII. Fig. 34. If $AB : ab :: 322 : 321$, the interval of the founds of thefe vibrations is $\frac{1}{4}$ comma nearly (*b*). Whence $321\ AB, = 322\,ab = AZ$, is the length of the fimple cycle of the diflocations of the pulfes of the vibrations AB, ab, or of the period of the imperfections of any confonances whofe vibrations are different multiples of AB and ab (*c*) and whofe temperament is the interval of the founds of AB and ab (*d*).

Now the vibrations of imperfect v^ths^ are AD and ac, or $3\,AB$ and $2\,ab$, and the two conftant lengths of their imperfect fhort cycles are $AG = 2\,AD = 2 \times 3\,AB$ and $ag = 3\,ac = 3 \times 2\,ab$.

Hence $AZ = 321\,AB = \frac{321}{3} \times 3\,AB = 107\ AD = \frac{321}{6} \times 6\,AB = 53\frac{3}{6}\,AG$;

Likewife $AZ = 322\,ab = \frac{322}{2} \times 2\,ab = 161\ ac = \frac{322}{6} \times 6\,ab = 53\frac{4}{6}\,ag$.

And after the coincidence of the pulfes, their firft diflocation is $gG = \frac{1}{161}\,AD$; and the limit of the greateft diflocation is $\frac{1}{2}\,ab = \frac{107}{644}\,AD$.

For

(*b*) Prop. XI. fchol. 4. art. 6.
(*c*) Prop. VIII.
(*d*) Prop. VIII. cor. 1.

For $AB : AB - Ab$, or $bB :: 322 : 1$, whence $Bb = \frac{1}{322} AB$, and $gG = 6bB = \frac{6AB}{322} = \frac{3AB}{161} = \frac{1}{161} AD$, and $\frac{1}{2} ab = \frac{1}{2} \times \frac{321}{322} AB = \frac{1}{2} \times \frac{321}{322} \times \frac{1}{3} AD = \frac{107}{644} AD$, and is the limit of the greateſt diſlocation, or alternate leſſer interval of the pulſes of AB, ab in any half period, by prop. VII.

Now by an experiment mentioned in prop. XVIII, I found that the particles of air in an organ pipe called *d* or *d-la-ſol-re*, in the middle of the ſcale of the open diapaſon, made 262 complete vibrations or returns to the places they went from, and conſequently propagated 262 pulſes of air to the ear (*e*) in one ſecond of time; though the pitch of the organ was above half a tone lower than the preſent pitch at the Opera. And taking that ſound for the baſe of our vth, whoſe vibration AD repreſents a certain quantity of time, we have $262 AD = 1$ ſecond and hence the abſolute times AZ, AG, Gg and $\frac{1}{2} ab$ are the following fractions of $1''$.

For, $262 AD : AZ$ or $107 AD :: 1'' : \frac{107}{262} \times 1''$.

and $262 AD : AG$ or $2 AD :: 1'' : \frac{1}{131} \times 1''$.

and $262 AD : Gg$ or $\frac{1}{161} AD :: 1'' : \frac{1}{262 \times 161} \times 1'' = \frac{1}{42182} \times 1''$.

and

(*e*) Sect. I. art. 12.

and $262\,AD : \frac{1}{2}\,ab$ or $\frac{107}{644}\,AD :: 1'' : \frac{107}{262 \times 644} \times 1'' = \frac{1}{1568} \times 1''$.

And the reciprocal of the periodical time AZ, $= \frac{107}{262} \times 1''$, is the number of periods and alſo of beats in $1''$ (*f*) namely $\frac{262}{107} = 2.45$ nearly in $1''$, or 245 in $100''$ nearly.

And the leaſt diſlocations in the ſhort cycles, as $\Delta\varepsilon\lambda K$, which include the ſucceſſive periodical points Z, are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$ of the diſlocation Gg next to the coincident pulſes.

And theſe meaſures are to thoſe in any other v^ths in the ſcale of that organ, in the given ratio of the times of the ſingle vibrations of their baſes. And the like meaſures in any other given conſonance, whoſe temperament is given, may be computed in the like manner, or derived from theſe by the corollaries to prop. IX.

In this example the v^ths were tempered ſharp, and when they are equally tempered flat, by taking *Ad* and *AC* for the ſingle vibrations, the computation and the meaſures will be but very little different.

(*f*) Prop. X.

Scholium

Scholium 4 *to prop. xx.*

Tables and obſervations on the numbers of beats of the concords in the principal ſyſtems.

The following table ſhews the number of beats made in 15 ſeconds, by the ſeveral concords to the baſe note D, or D-ſol-re, at the Roman Pitch, and likewiſe the proportions of the beats of the ſame concords to any other baſe note; with this deſign, that perſons wanting leiſure or proper qualifications for examining the principles and concluſions requiſite to determine the ſyſtem of equal harmony, may yet form ſome judgment of its advantages and diſadvantages, when compared with the ſyſtem of mean tones and that of Mr. *Huygens*; upon this allowed principle, that, *cæteris paribus*, concords to the ſame baſe are more or leſs diſagreeable in their kind for beating faſter or ſlower reſpectively.

To aſſiſt the reader's judgment I have added the following obſervations reſulting from inſpection of the firſt table.

1. The beats in column 1 differ leſs from one another than thoſe in column 2 and 3 do, agreeably to the Name of the ſyſtem.

2. The beats of the V, V + VIII, V + 2VIII in col. 1, are a little quicker than thoſe in col. 2, in the ratio of 10 to 9, and quicker than thoſe in col. 3, in a ratio ſomething greater. But the

beats of the VI, VI + VIII, VI + 2VIII in col. 1 are much ſlower than thoſe in col. 2 and 3, in the ratio of 2 to 3 and more.

3. Theſe quick beats of the VI[th] in col. 2 and 3 have the diſadvantage to be doubled in every

TAB. I.

The beats in 15″ *of all the* - - - - - -

VI + 2VIII. $\frac{3}{20}$	80	120	$133\frac{1}{11}$
V + 2VIII. $\frac{1}{6}$	40	36	$34\frac{8}{11}$
III + 2VIII. $\frac{1}{5}$	13	0	$4\frac{1}{4}$
VI + VIII. $\frac{3}{10}$	40	60	$66\frac{6}{11}$
V + VIII. $\frac{1}{3}$	20	18	$17\frac{4}{11}$
III + VIII. $\frac{2}{5}$	13	0	$4\frac{1}{4}$
VI. $\frac{3}{5}$	20	30	$33\frac{3}{11}$
V. $\frac{2}{3}$	20	18	$17\frac{4}{11}$
III. $\frac{4}{5}$	13	0	$4\frac{1}{4}$
Syſtem of	equal harm.	mean tones	M. *Huygens.*
Column	1	2	3

every superior octave, whereas the quick beats of the v^{th} in col. 1. remain the same in the second octave and are only double in the third &c.

4. The beats of the III^{d} and compounds are quicker indeed in col. 1 than those in col. 2 and

TAB. I.

- - - - - *concords to the base* D-sol-re.

3^{d} + 2VIII. $\frac{5}{24}$	96	144	160
4^{th} + 2VIII. $\frac{3}{16}$	107	96	93
6^{th} + 2VIII. $\frac{5}{32}$	83	0	27
3^{d} + VIII. $\frac{5}{12}$	48	72	80
4^{th} + VIII. $\frac{3}{8}$	53	48	46
6^{th} + VIII. $\frac{5}{16}$	42	0	14
3^{d}. $\frac{5}{6}$	24	36	40
4^{th}. $\frac{3}{4}$	$26\frac{2}{3}$	24	23
6^{th}. $\frac{5}{8}$	$20\frac{4}{5}$	0	7
System of	equal harm.	mean tones	M. *Huygens*.
Column	1	2	3

3; but being ſlower than the beats of the other concords in all the ſyſtems, they can ſcarce be ſo offenſive as theſe will be.

5. Likewiſe the beats of the 6th and compounds in col. 1, being ſlower than thoſe of the 4th

TAB. II.

The order of the harmony - - - - - -

VI + 2VIII	240	360	$399\frac{3}{11}$
V + 2VIII	40	36	$34\frac{8}{11}$
III + 2VIII	13	0	$4\frac{1}{4}$
VI + VIII	120	180	$199\frac{7}{11}$
V + VIII	20	18	$17\frac{4}{11}$
III + VIII	26	0	$8\frac{1}{2}$
VI	60	90	$99\frac{9}{11}$
V	40	36	$34\frac{8}{11}$
III	52	0	17
Syſtem of	equal harm.	mean tones	M. *Huygens*.
Column	1	2	3

4^{th} and 3^{d} and compounds with the ſame number of VIII^{ths} in all the ſyſtems, can hardly offend the ear ſo much as the quicker beats of theſe other concords will.

6. The ſums of the beats of all the concords to

TAB. II.

- - - - - - - - - *of all the concords.*

3^{d} + 2VIII	480	720	800
4^{th} + 2VIII	321	288	279
6^{th} + 2VIII	415	0	135
3^{d} + VIII	240	360	400
4^{th} + VIII	159	144	138
6^{th} + VIII	210	0	70
3^{d}	120	180	200
4^{th}	80	72	69
6^{th}	104	0	35
Syſtem of	equal harm.	mean tones	M. *Huygens.*
Column	1	2	3

to the note D-ſol-re in both the col. 1, 2, 3, are reſpectively 759, 702, 805, whoſe proportions with reſpect to the ſame or any other baſe note are 38, 35, 40 very nearly. The ſmall exceſs of the firſt ſum above the ſecond ariſes chiefly from the beats of the III^{d} and 6th with their compounds, which in all probability are inoffenſive, as we ſaid before.

But a completer rule for comparing the harmony of imperfect concords to a given baſe, appears in the ſecond table; *That concord to which a ſmaller number correſponds, being more harmonious, in its kind, than any other to which a larger number correſponds*; which affords two or three more obſervations.

7. In col. 1 and 3, the III^{ds} become more harmonious by the addition of VIII^{ths}.

8. In all the ſyſtems, the V + VIII is more harmonious than the V and V + 2VIII.

9. In col. 1, the Vth is more harmonious than the III^{d} and VIth, the V + VIII than the III + VIII and VI + VIII, and likewiſe the 4th and compounds than the 6th and 3^{d} and compounds with equal numbers of VIII^{ths}.

10. It may be objected to the ſyſtem of equal harmony that the beats of the V^{ths} are not only a little quicker, but ſomething ſtronger and diſtincter than thoſe of the other concords; which deſerves to be conſidered. On the other hand it ſhould be conſidered too, whether thoſe very quick and leſs diſtinct beats of the VIth and compounds,

pounds, have not a worſe effect in deſtroying the clearneſs of their harmony.

Theſe are the principal advantages and diſadvantages that occur in comparing theſe ſyſtems. For as to the falſe concords being ſomething worſe in the ſyſtem of equal harmony than in the other two (*g*), this is no objection to the ſyſtem, but only to the application of it to defective inſtruments; and I have ſhewn above how to ſupply their defects, without the leaſt inconvenience to the performer (*h*).

I ſhall only obſerve, that the firſt table was calculated from the temperaments of the ſyſtems in prop. XVII and ſcholium, by the corollaries to prop. XI; and that the numbers in that table multiplied by the numerators of the known fractions, annexed to the characters of the correſponding concords, produce the correſponding numbers in the ſecond table, according to coroll. 12, prop. XIII.

Scholium 5 to prop. xx.

The ſound of an open metalline pipe will be flattened or ſharpened a little by bending a ſmall part of the metal at the open end, a little inwards or outwards, reſpectively.

The ſound of a ſtopt pipe, made of metal, will be flattened or ſharpened a little by bending the ears, at the ſides of the mouth, a little inwards or outwards, reſpectively.

The

(*g*) Sect. VIII. art. 2. (*h*) Sect. VIII. art. 11th.

The ſound of an open wooden pipe will be flattened or ſharpened a little, by depreſſing or raiſing the leaden plate that hangs over the open end, reſpectively.

The ſound of a ſtopt wooden pipe will be flattened or ſharpened by drawing the plug outwards or forcing it inwards, reſpectively.

The ſound of a reed-pipe will be flattened or ſharpened by cauſing the brazen tongue to be lengthened or ſhortened, reſpectively.

There is ſomething curious in the reaſons of theſe effects, but as they cannot be well explained in few words, I muſt omit them.

A N

ALPHABETICAL INDEX,

referring first to the Pages by the common numbers, to the Propositions by the Roman numbers and to the Notes by the letters in books.

A.

B.

T the

C.

are

D.

E.

INDEX.

F.

G.

H.

Har-

I.

K.

L.

M.

N.

O.

the

the

S.

Tem-

FINIS.

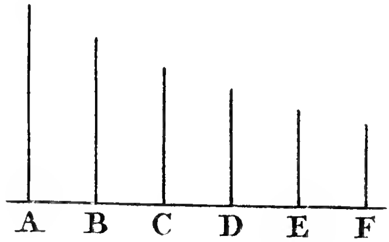
Corrections.

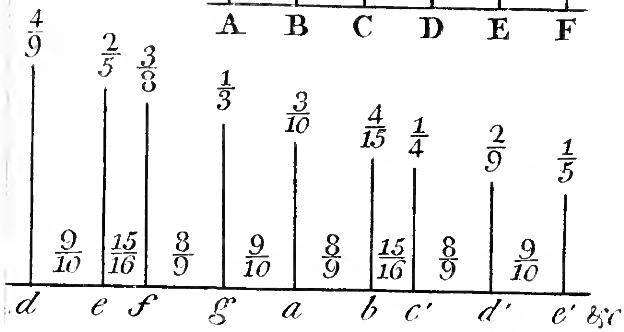
Page 6. note, lin. 10, for ὐξυτέρων, r. ὀξυτέρων.
11. lin. 4, for *cé*, r. *cc'*.
51. note, lin. 2, *dele* the first *minus*—
69, 71, 73 in the running titles, for Prop. VIII. r. Lemma.
71. lin. penult. for $-\frac{q}{p}a$, r. $-\frac{q}{p}d$.
114, 115, notes (*p*) and (*q*), for cor. 7. r. cor. 8.
168. lin. 1. *dele* 9.
169. lin. 12. for single, r. one.
163, 165, 167, 169, 171, in the running titles, for Prop. XVII. r. Art. 5, 7, 9, 12, 15, respectively.
227. note (*k*) for schol. r. schol. 4.

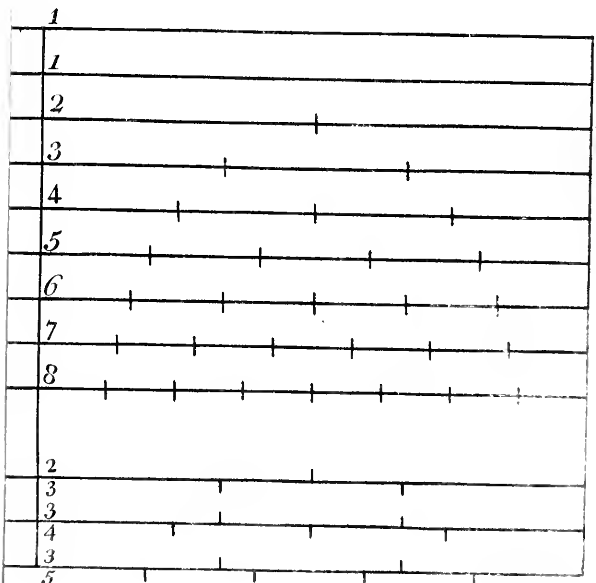
Pl. I.

Fig. 2

Fig. 1







W. Stephens fec

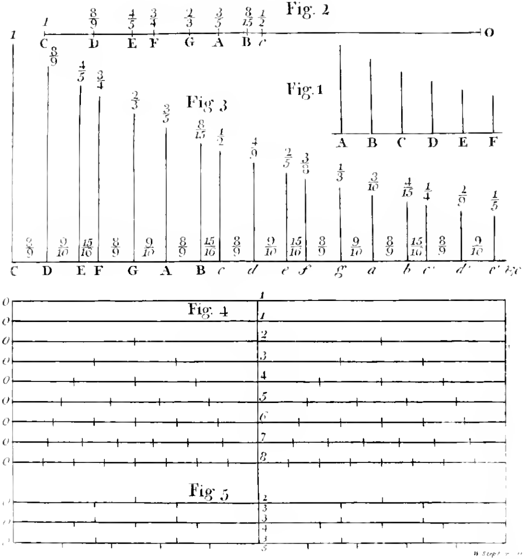
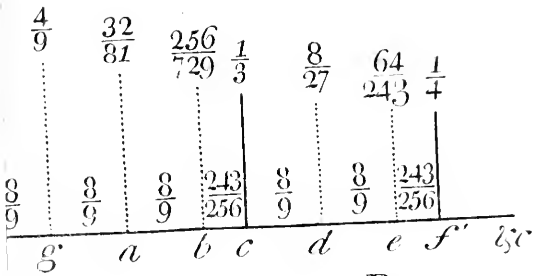
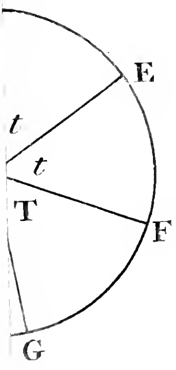
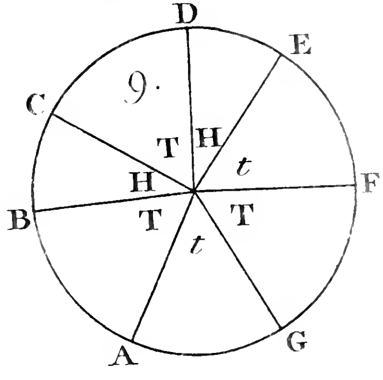


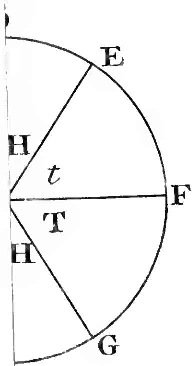
Fig. 2
1 8/9 4/5 3/4 2/3 3/5 8/15 1/2
C D E F G A B c O
Fig. 1
A B C D E F
Fig. 3
1 8/9 4/5 3/4 2/3 3/5 8/15 1/2 4/9 2/5 3/8 1/3 3/10 4/15 1/4 2/9 1/5
8/9 9/10 15/16 8/9 9/10 8/9 15/16 8/9 9/10 15/16 8/9 9/10 8/9 15/16 8/9 9/10
C D E F G A B c d e f g a b c' d' e' &c
Fig. 4
1 2 3 4 5 6 7 8
Fig. 5
2/3 3/4 3/5

Fig. 6.


4/9
32/81
256/729
1/3
8/27
64/243
1/4
8/9
8/9
8/9
243/256
8/9
8/9
243/256
g
a
b
c
d
e
f
&c


E
F
G
t
t
T


9.
D
E
F
G
A
B
C
H
T
H
t
T
T
t


E
F
G
H
t
T
H

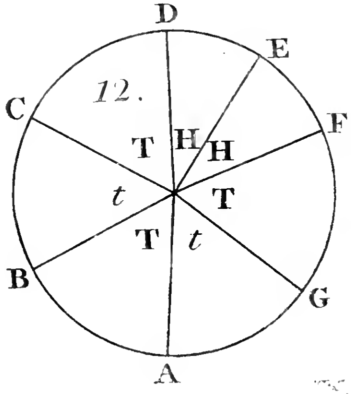

12.
D
E
F
G
A
B
C
T
H
H
t
T
T
t

Fig. 6.

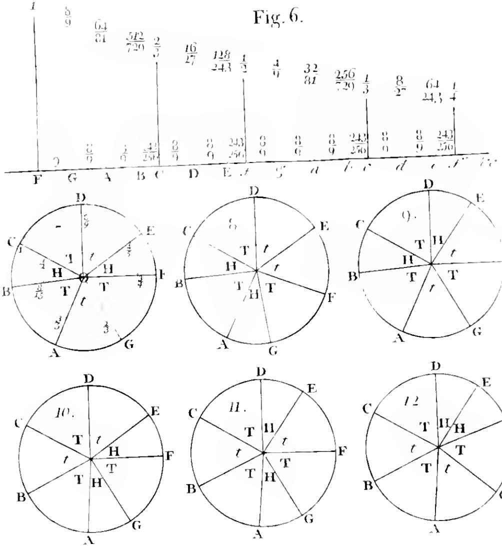
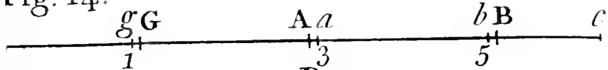


Fig. 14.



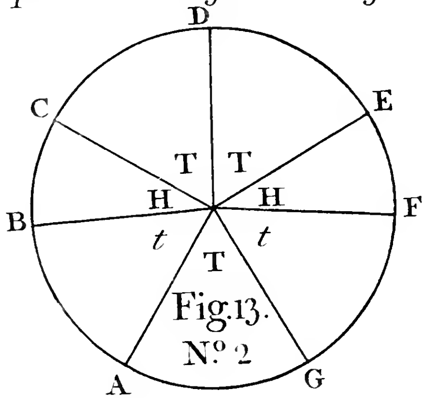
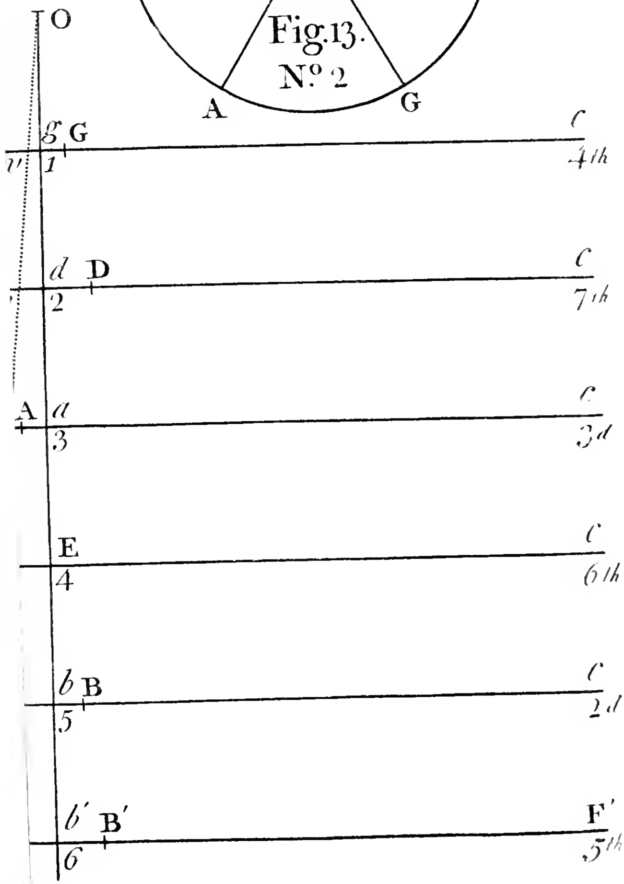
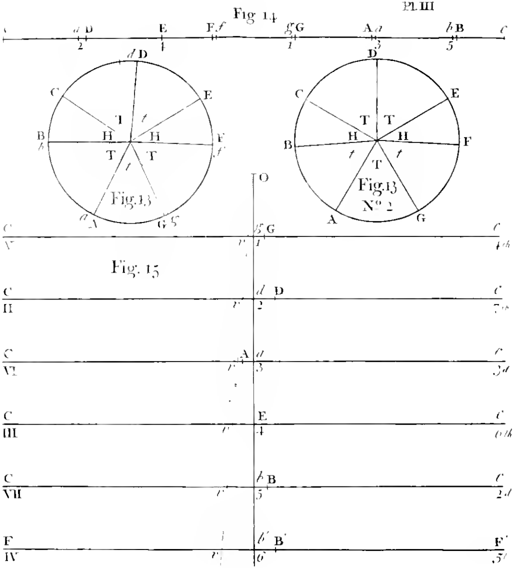


Fig. 13.
N° 2



Pl. III



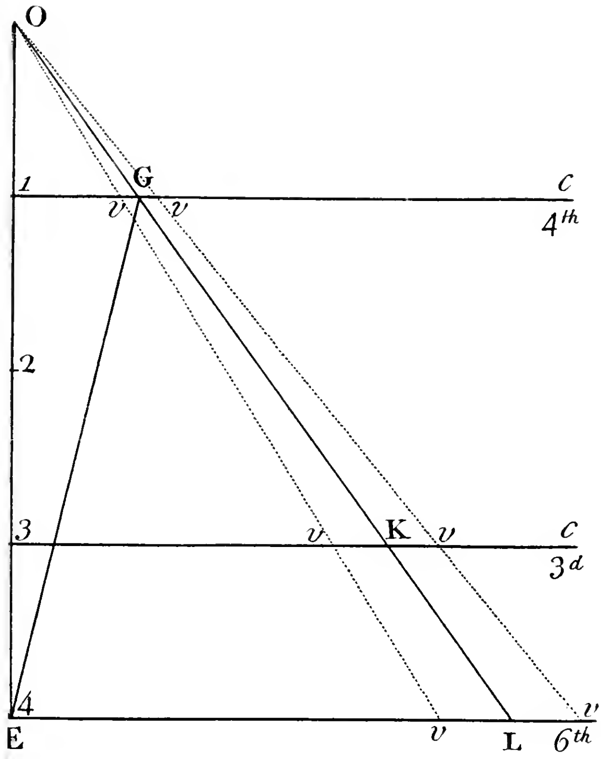

O
1
G
c
v
v
4th
2
3
v
K
v
c
3d
4
E
v
L
v
6th

Fig. 16.

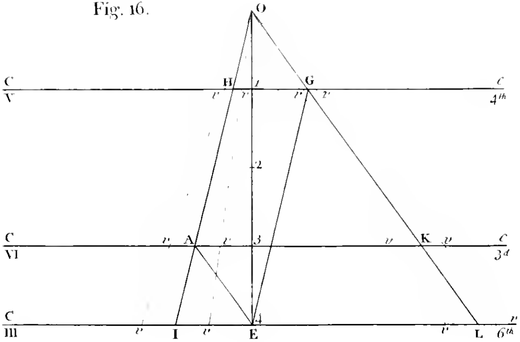
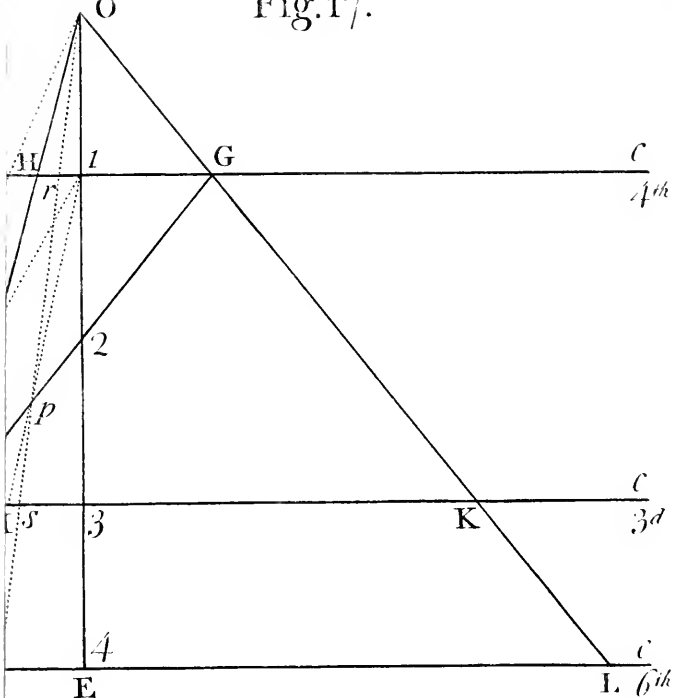
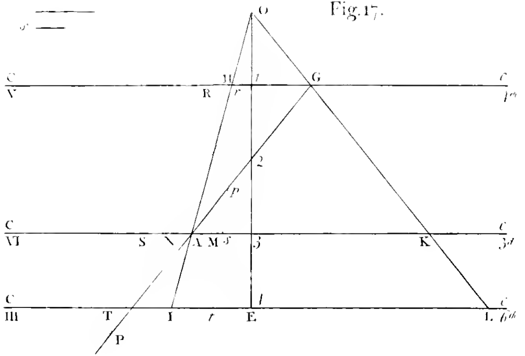


Fig. 17.



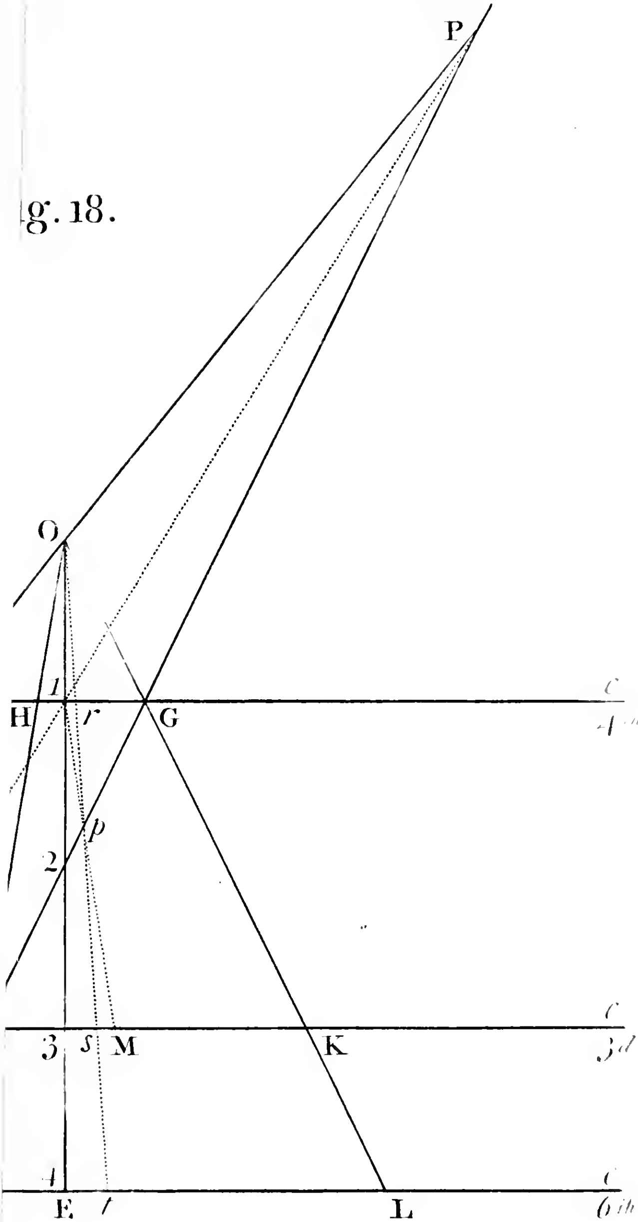
H.S. sc.

Fig. 17.



Pl.VI.

ig. 18.



PL.VI.

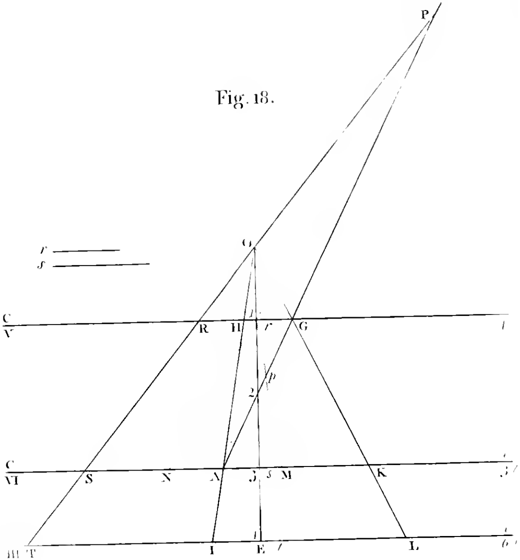
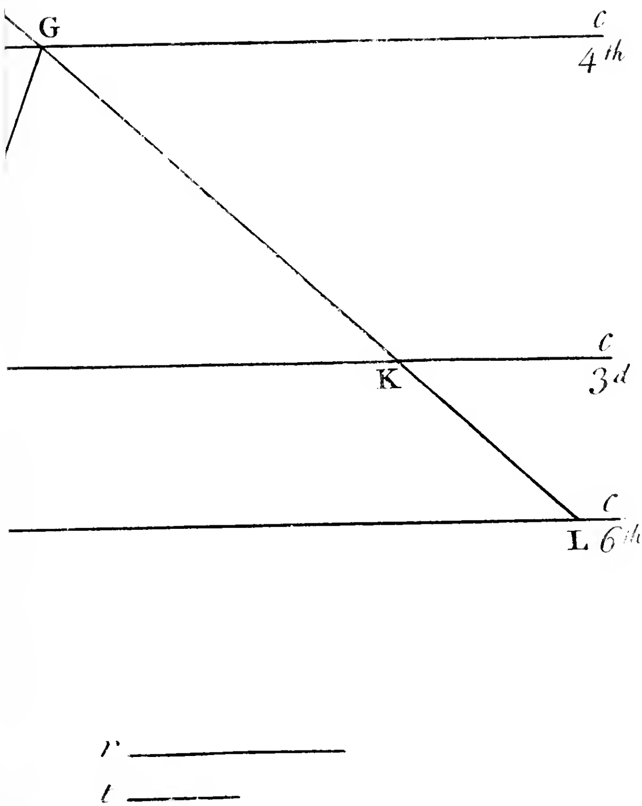


Fig. 18.

Fig. 19.



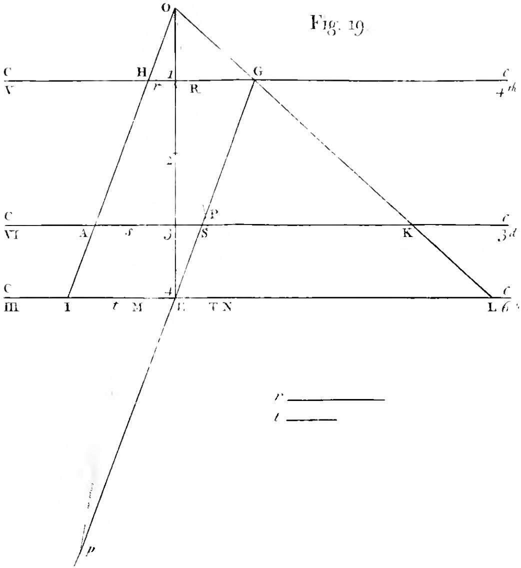


Fig. 19.

Fig. 20.

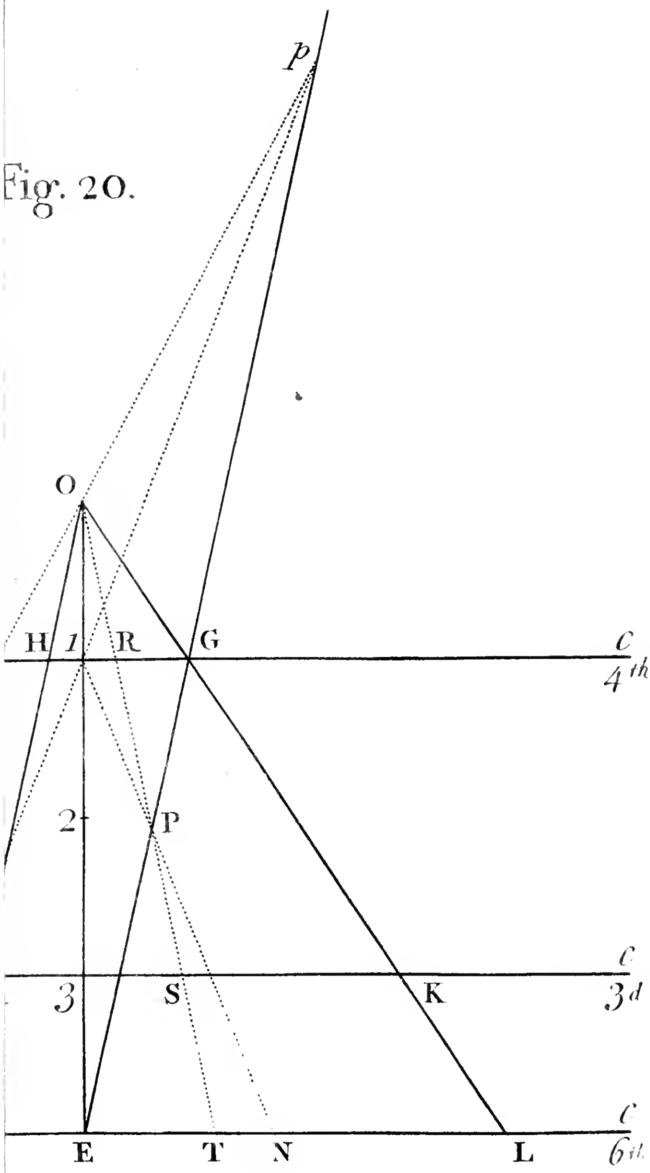


Fig. 20.

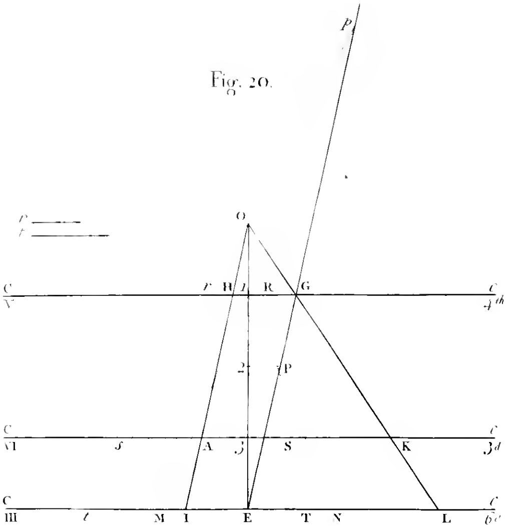
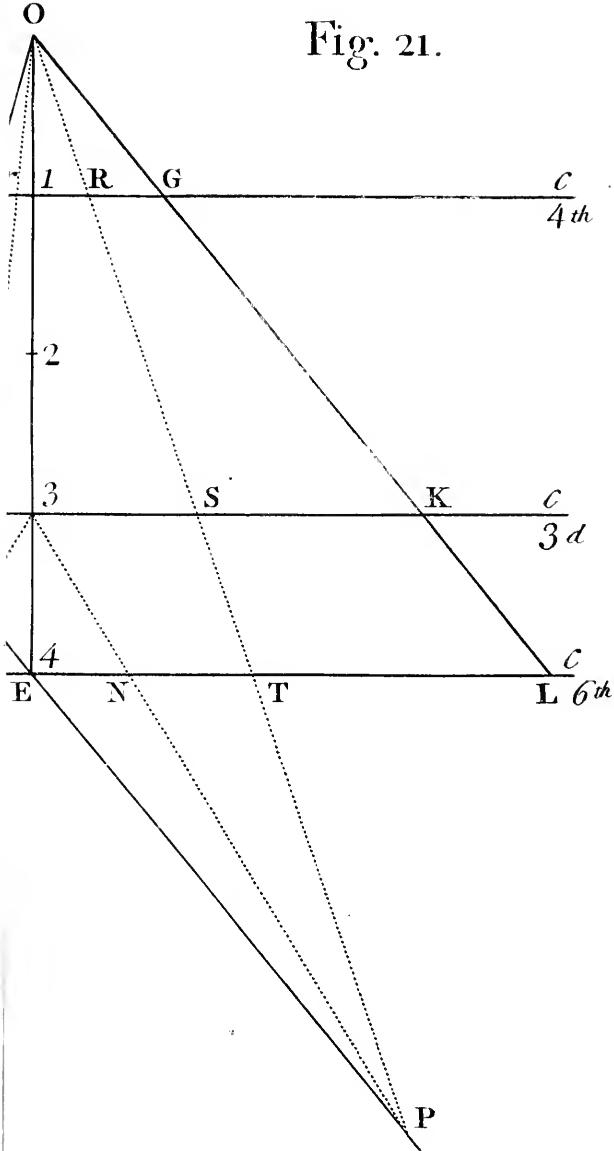


Fig. 21.



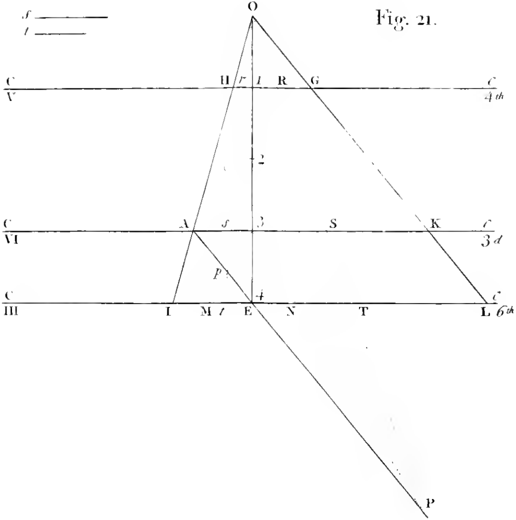
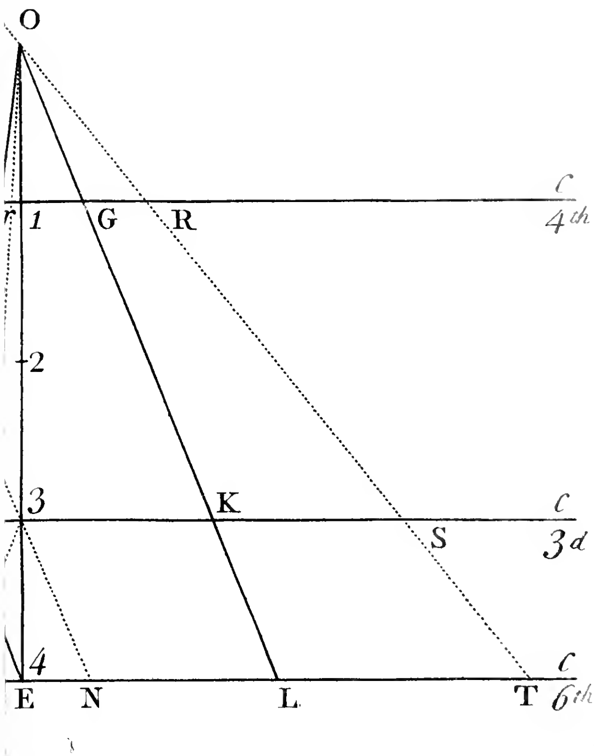


Fig. 21.

ig. 22.



Pl. X.

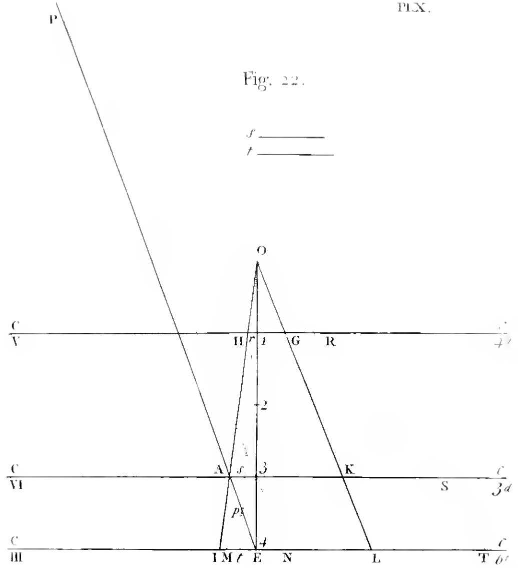
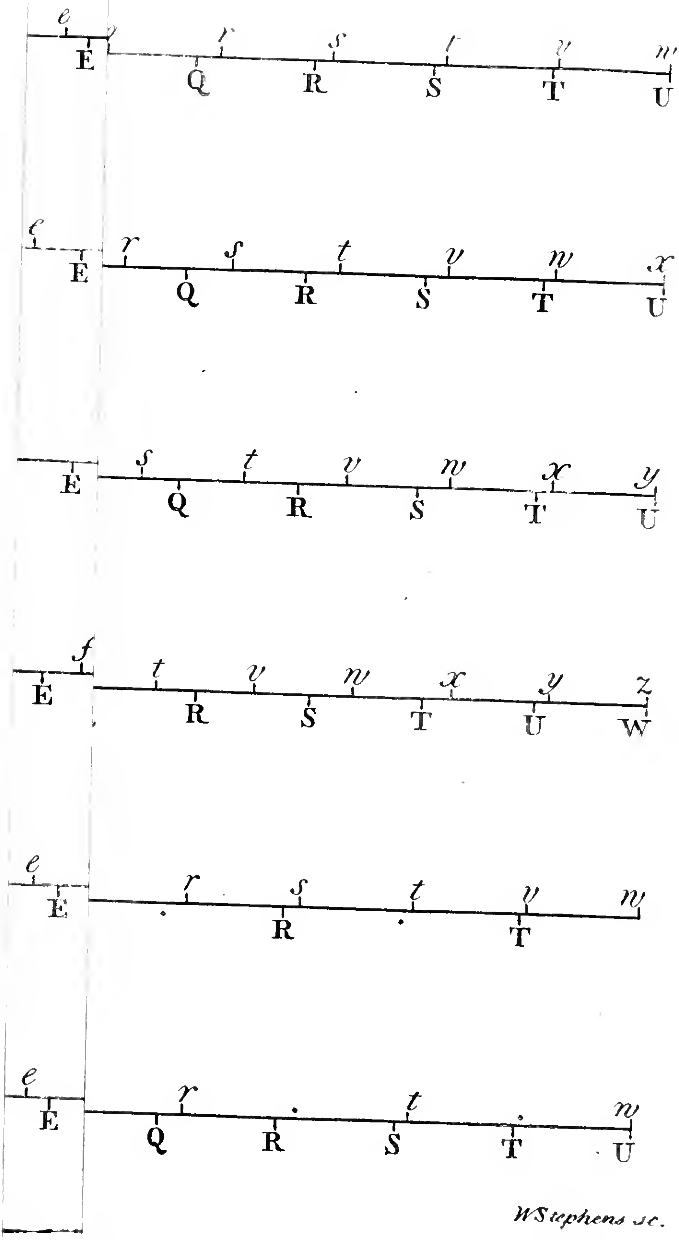
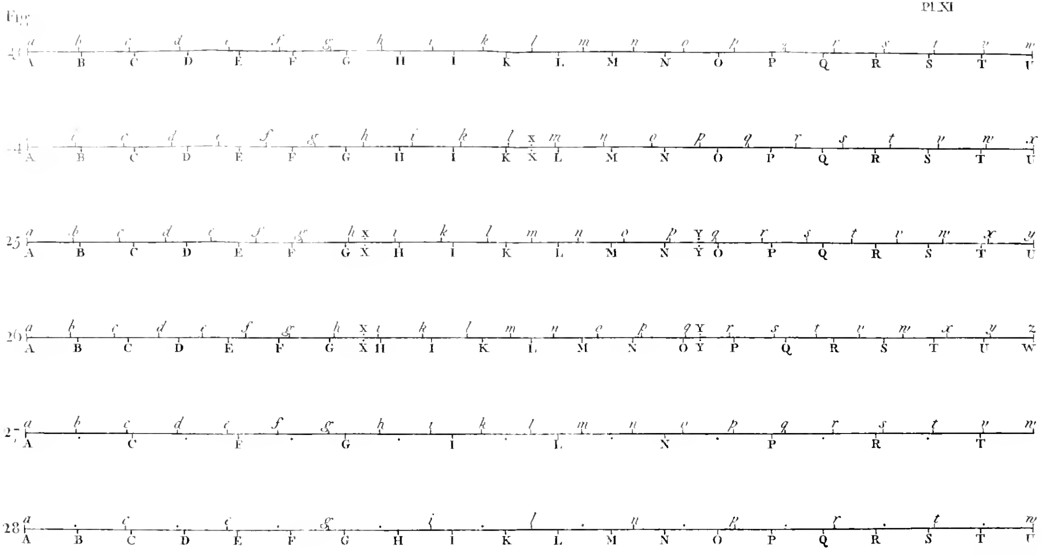


Fig. 22.

Pl. XI.



W Stephens sc.



H Stephens sc.

Plate XII.

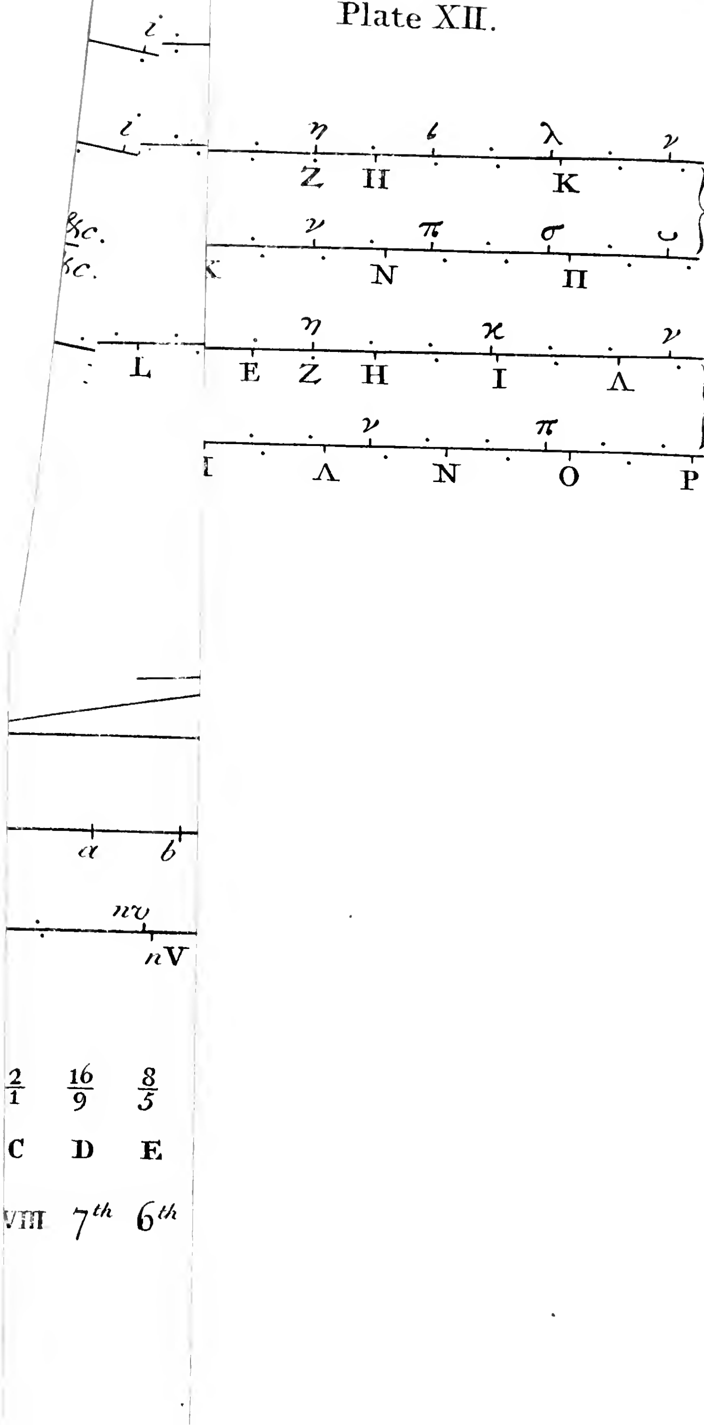


Plate XII.

33, 34, 35

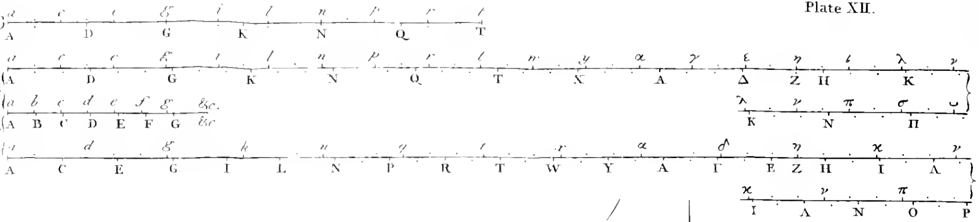
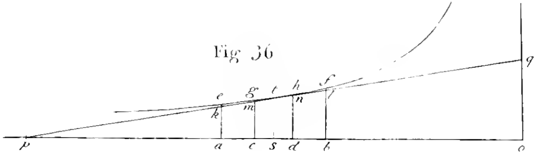
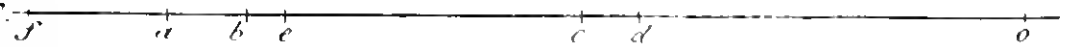


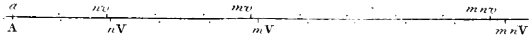
Fig 36



37.

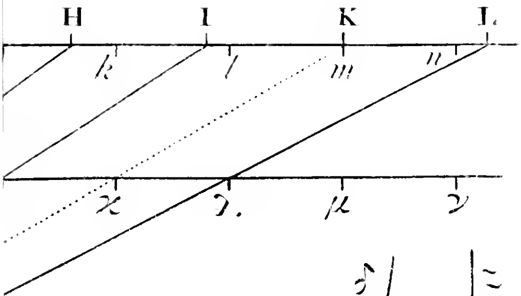


38.



Tab 1.

$\frac{2}{1}$	$\frac{16}{9}$	$\frac{8}{5}$	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{6}{5}$	$\frac{16}{15}$	1	$\frac{8}{9}$	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{8}{15}$	$\frac{1}{2}$	&c.
C	D	E	F	G	A	B	*c*	*d*	*e*	*f*	*g*	*a*	*b*	*c'*	
VIII	7th	6th	V	4th	3d	2d	γ	II	III	4th	V	VI	VII	8th	



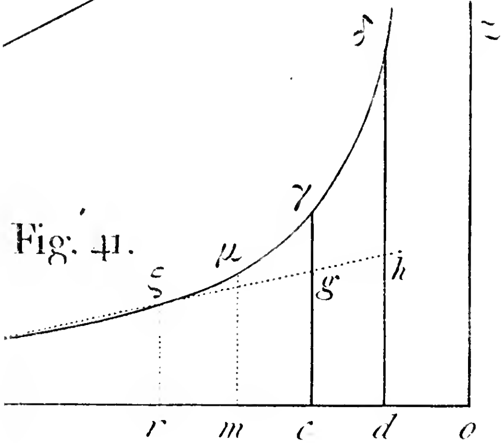


Fig. 41.

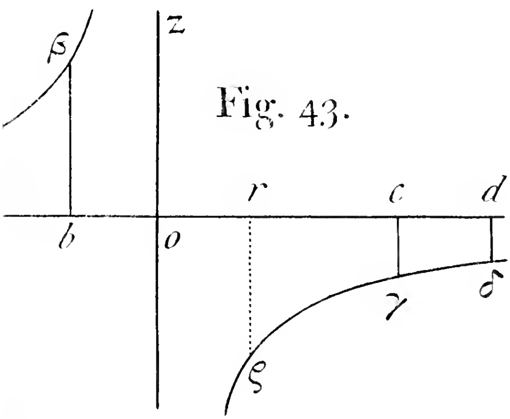


Fig. 43.

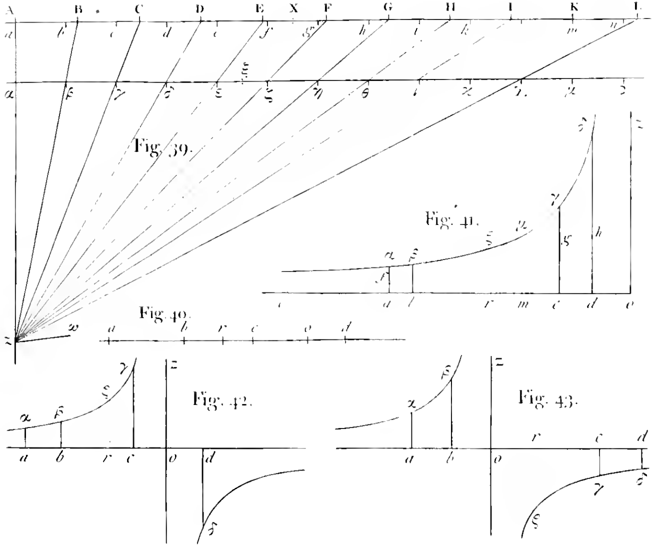


Fig. 39.

Fig. 40.

Fig. 41.

Fig. 42.

Fig. 43.

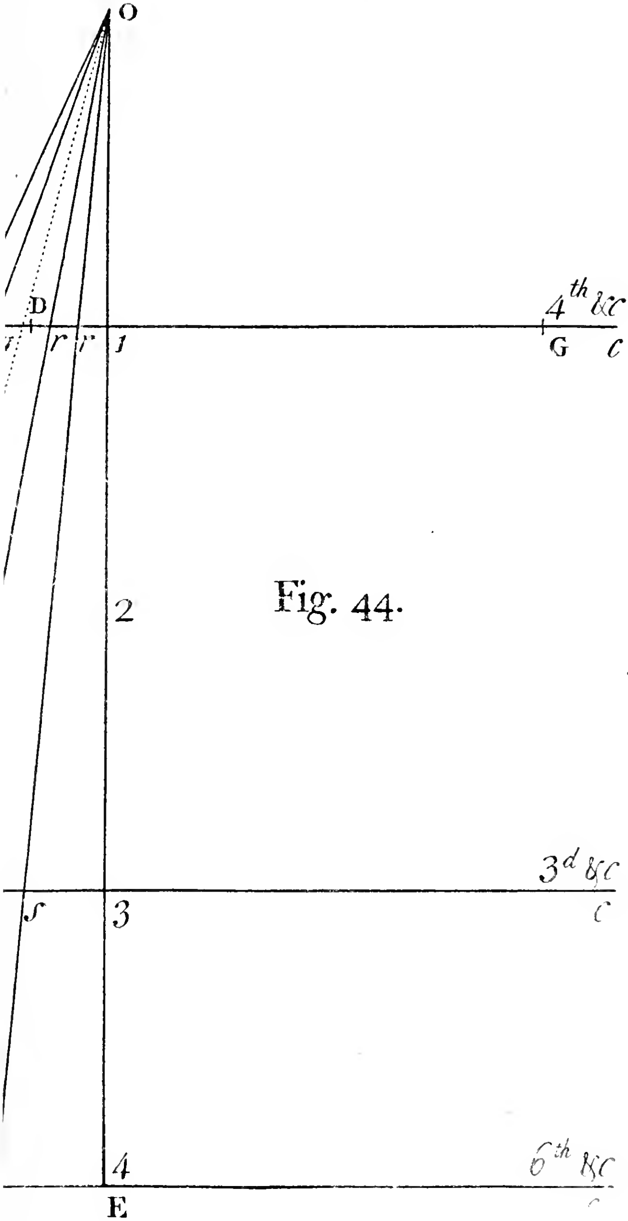


Fig. 44.

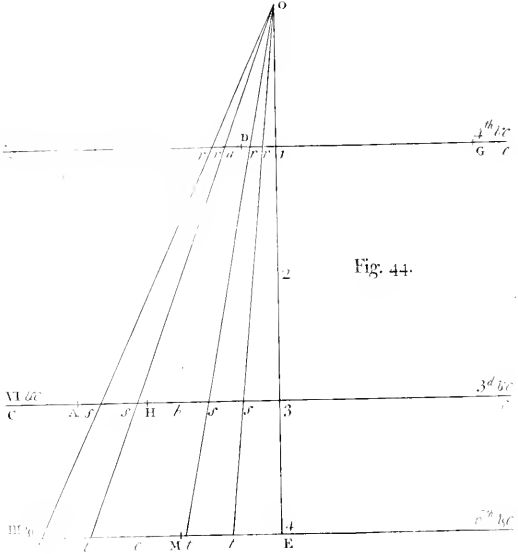


Fig. 44.

Pl.XV.

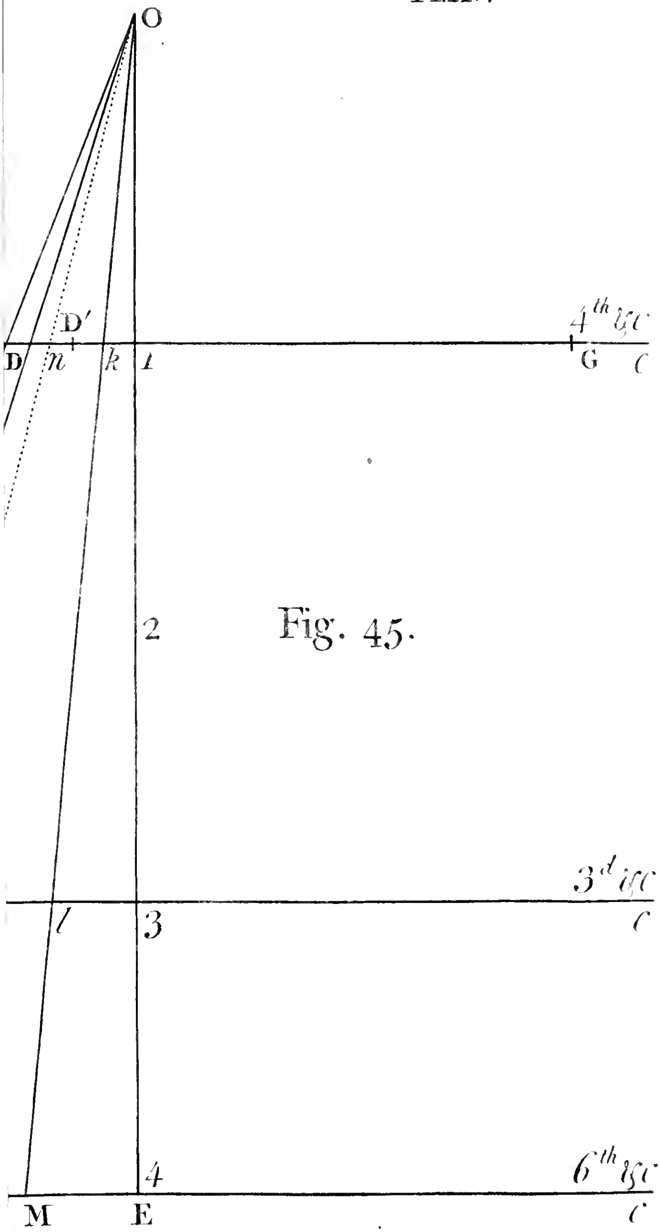


Fig. 45.

Pl XV

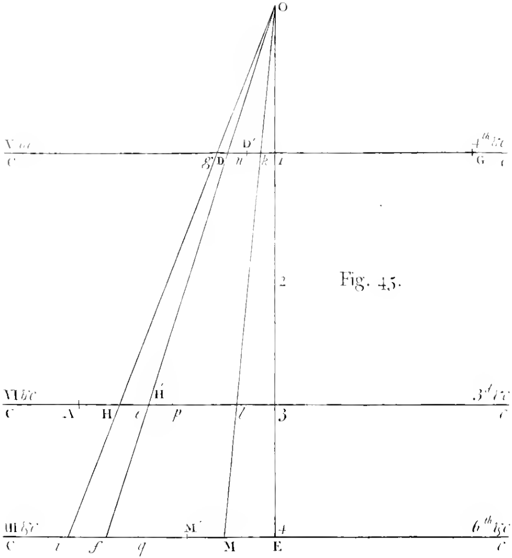


Fig. 45.

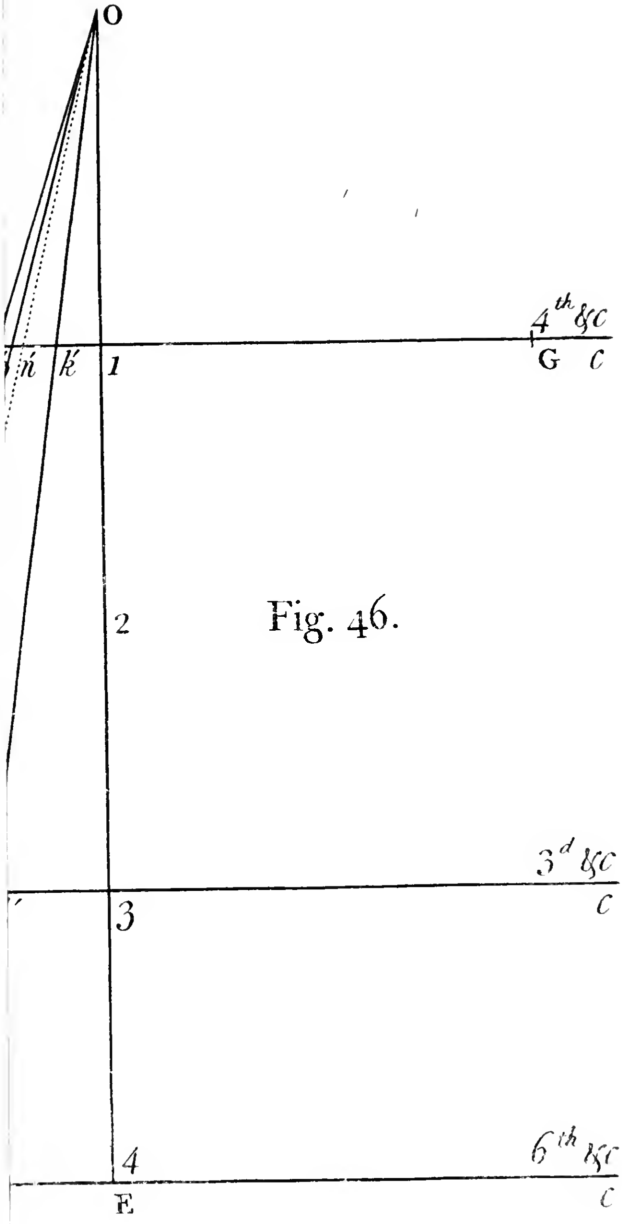


Fig. 46.

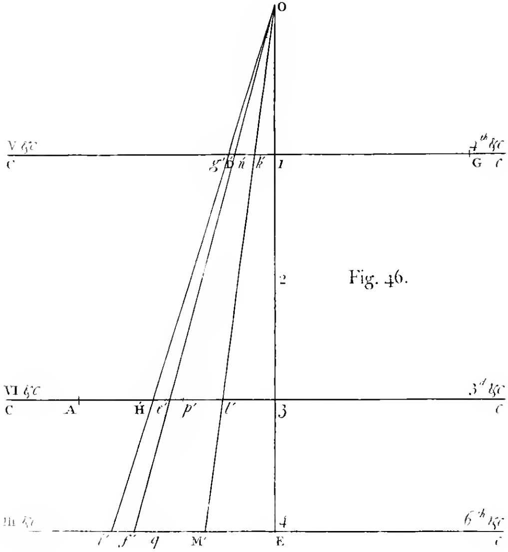


Fig. 46.

Pl XVII

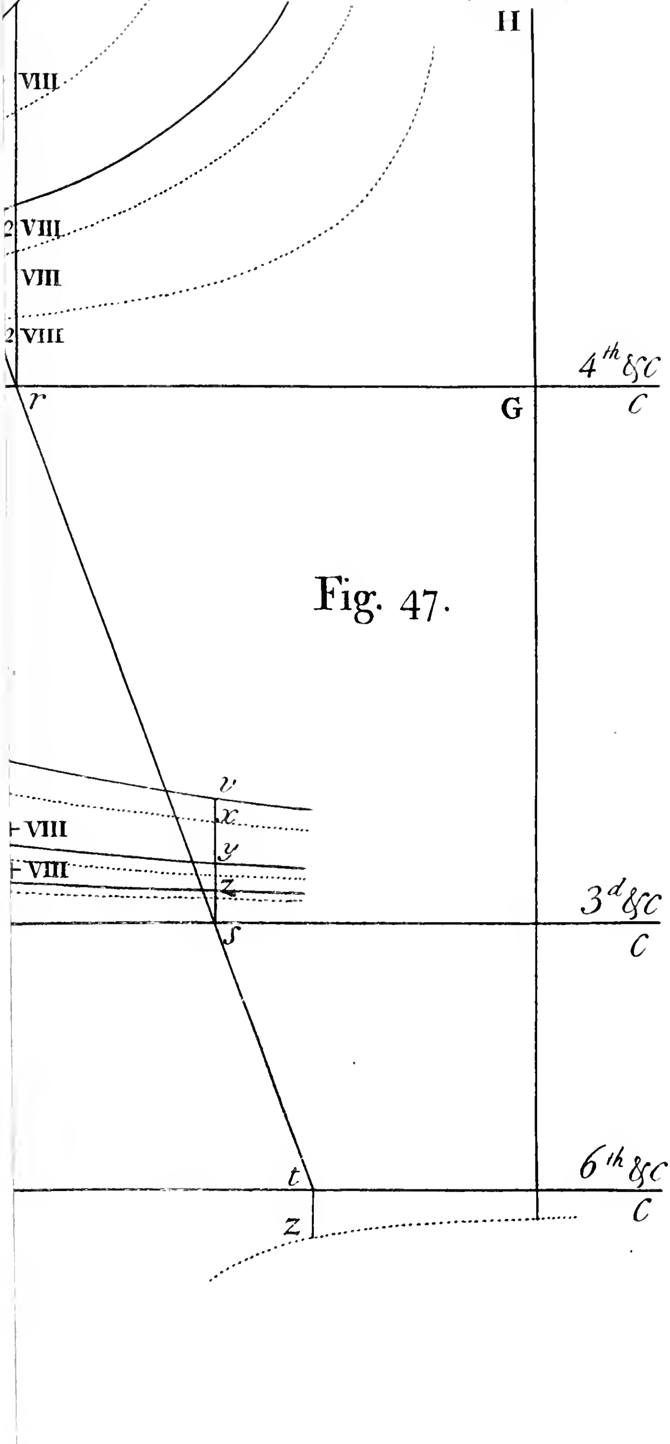


Fig. 47.

Pl. XVII

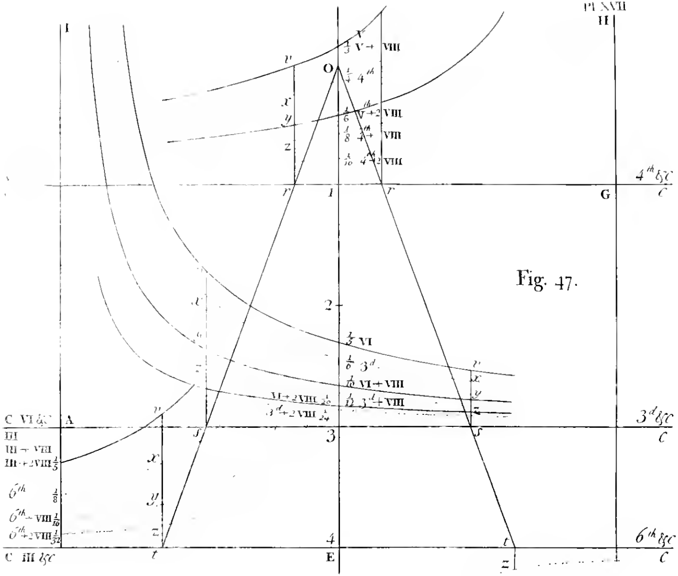
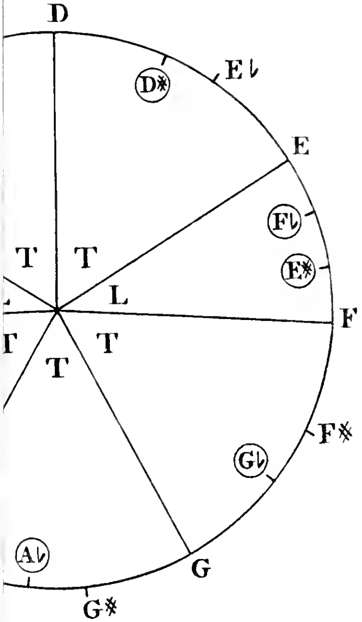


Fig. 47.


D
D♯
E♭
E
F♭
E♯
F
F♯
G♭
G
A♭
G♯
T
T
L
T
T
T

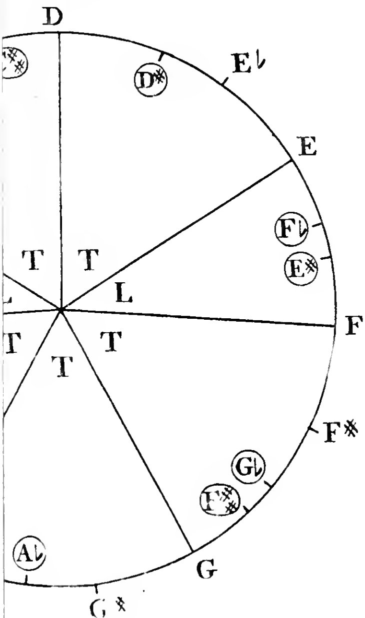

D
D♯
E♭
E
F♭
E♯
F
F♯
G♭
G
A♭
G♯
T
T
L
T
T
T

Fig. 48.

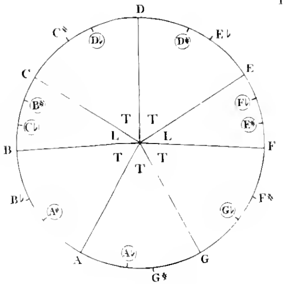
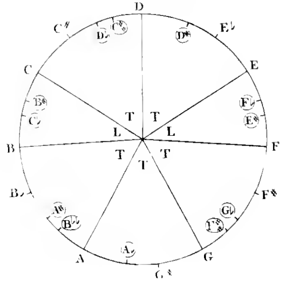

D
C♯
D♭
D♯
E♭
E
C
B♯
C♭
F♭
E♯
T
T
L
L
B
F
T
T
T
B♭
A♯
G♭
F♯
A♭
A
G
G♯

Fig. 49.


D
C♯
D♭
D♯
E♭
E
C
B♯
C♭
F♭
E♯
T
T
L
L
B
F
T
T
T
B♭
A♯
B♭♭
G♭
F♯
F♯
F♯
A♭
A
G
G♯

r & major Consonances to 24 Keys. Pl. XIX

4th	IV	5th	V	6th	VI	7th	VII	8th
F**	·	G*	·	A*	·	B*	·	c**
B*	·	C*	C**	D*	·	E*	·	f**
E*	·	F*	F**	G*	·	A*	·	b*
A*	·	B	B*	C*	C**	D*	·	e*
D*	·	E	E*	F*	F**	G*	·	a*
G*	·	A	A*	B	B*	C*	C**	d*
C*	C**	D	D*	E	E*	F*	F**	g*
F*	F**	G	G*	A	A*	B	B*	c*
B	B*	C	C*	D	D*	E	E*	f*
E	E*	F	F*	G	G*	A	A*	b
A	A*	B♭	B	C	C*	D	D*	e
D	D*	E♭	E	F	F*	G	G*	a
G	G*	A♭	A	B♭	B	C	C*	d
C	C*	D♭	D	E♭	E	F	F*	g
F	F*	G♭	G	A♭	A	B♭	B	c
B♭	B	C♭	C	D♭	D	E♭	E	f
E♭	E	F♭	F	G♭	G	A♭	A	b♭
A♭	A	B♭♭	B♭	C♭	C	D♭	D	e♭
D♭	D	·	E♭	F♭	F	G♭	G	a♭
G♭	G	·	A♭	B♭♭	B♭	C♭	C	d♭
C♭	C	·	D♭	·	E♭	F♭	F	g♭
F♭	F	·	G♭	·	A♭	B♭♭	B♭	c♭
B♭♭	B♭	·	C♭	·	D♭	·	E♭	f♭
·	E♭	·	F♭	·	G♭	·	A♭	b♭
1	D	1	L	1	L	1	L	

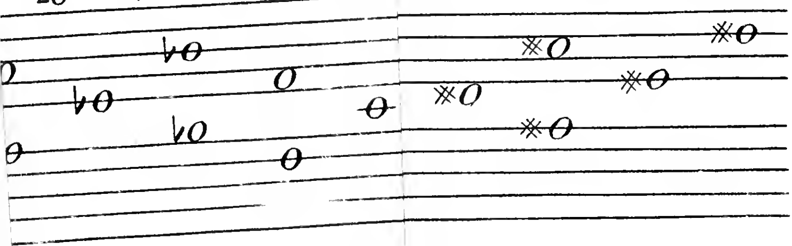
A Table of minor & major Consonances to 24 Keys. Pl XIX

Key	2d	IId	3d	III	4th	IV	5th	V	6th	VI	7th	VII	8th
C♯	D♯	·	E♯	·	F♯	·	G♯	·	A♯	·	B♯	·	c♯
F♯	G♯	·	A♯	·	B♯	·	C♯	C♯	D♯	·	E♯	·	f♯
B♯	C♯	C♯	D♯	·	E♯	·	F♯	F♯	G♯	·	A♯	·	b♯
E♯	F♯	F♯	G♯	·	A♯	·	B	B♯	C♯	C♯	D♯	·	e♯
A♯	B	B♯	C♯	C♯	D♯	·	E	E♯	F♯	F♯	G♯	·	a♯
D♯	E	E♯	F♯	F♯	G♯	·	A	A♯	B	B♯	C♯	C♯	d♯
G♯	A	A♯	B	B♯	C♯	C♯	D	D♯	E	E♯	F♯	F♯	g♯
C♯	D	D♯	E	E♯	F♯	F♯	G	G♯	A	A♯	B	B♯	c♯
F♯	G	G♯	A	A♯	B	B♯	C	C♯	D	D♯	E	E♯	f♯
B	C	C♯	D	D♯	E	E♯	F	F♯	G	G♯	A	A♯	b
E	F	F♯	G	G♯	A	A♯	B♭	B	C	C♯	D	D♯	e
A	B♭	B	C	C♯	D	D♯	E♭	E	F	F♯	G	G♯	a
D	E♭	E	F	F♯	G	G♯	A♭	A	B♭	B	C	C♯	d
G	A♭	A	B♭	B	C	C♯	D♭	D	E♭	E	F	F♯	g
C	D♭	D	E♭	E	F	F♯	G♭	G	A♭	A	B♭	B	c
F	G♭	G	A♭	A	B♭	B	C♭	C	D♭	D	E♭	E	f
B♭	C♭	C	D♭	D	E♭	E	F♭	F	G♭	G	A♭	A	b♭
E♭	F♭	F	G♭	G	A♭	A	B♭	B♭	C♭	C	D♭	D	e♭
A♭	B♭	B♭	C♭	C	D♭	D	·	E♭	F♭	F	G♭	G	a♭
D♭	·	E♭	F♭	F	G♭	G	·	A♭	B♭	B♭	C♭	C	d♭
G♭	·	A♭	B♭	B♭	C♭	C	·	D♭	·	E♭	F♭	F	g♭
C♭	·	D♭	·	E♭	F♭	F	·	G♭	·	A♭	B♭	B♭	c♭
F♭	·	G♭		A♭	B♭	B♭	·	C♭	·	D♭	·	E♭	f♭
B♭		C♭	·	D♭	·	E♭	·	F♭	·	G♭	·	[illegible]	b♭
	L	l	L	l	L	l	D	l	L	l	L	l	L

Tab. Pl. XX.

...of beats in 15 Seconds, o...

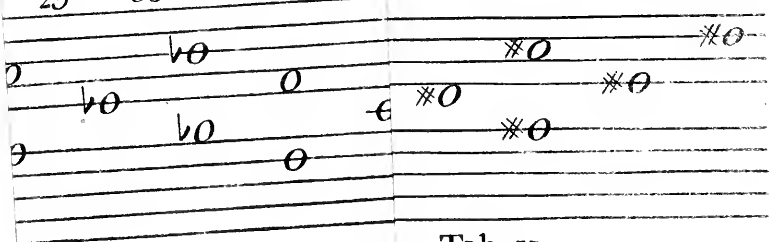
33	49	36	27	32	47	35	53
32	47	35	26	31	46	34	52
31	46	34	26	30	45	34	50
30	45	34	25	29	44	33	48
29	44	33	24	28	42	32	47
28	42	32	24	28	41	31	46



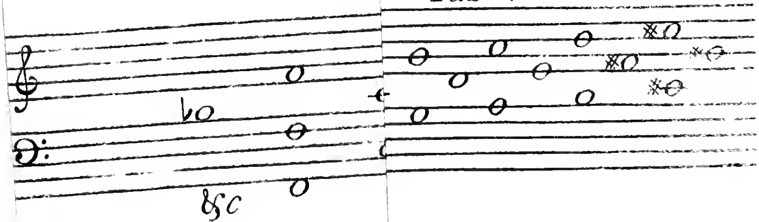
Tab.

...of beats in 15 Second...

29	44	33	24	29	43	32	48
28	43	32	24	28	42	31	47
28	41	31	23	27	40	30	45
27	40	30	23	26	39	29	44
26	39	29	22	26	38	29	43
25	38	28	21	25	37	28	42



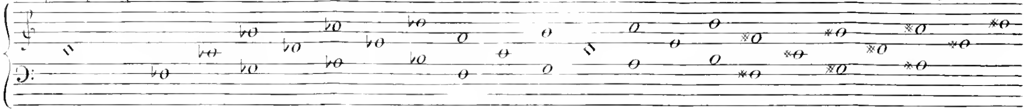
Tab. v.



Tab II

Pitch. ¼ tones. The numbers of beats in 15 Seconds, of the Vths in the System of equal harmony.

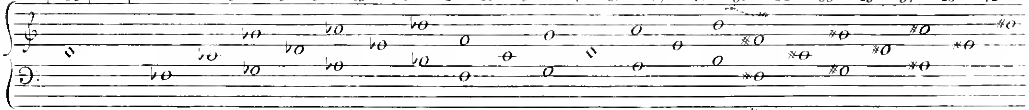
Pitch	¼ tones																				
293	4	26	38	29	43	33	40	36	27	41	30	45	34	50	38	28	42	32	47	35	53
285	3	26	38	28	41	32	47	35	26	40	30	44	33	49	37	28	41	31	46	34	52
277	2	25	37	28	40	31	46	34	26	39	29	43	32	48	36	27	40	30	45	34	50
269	1	24	36	27	39	30	45	34	25	37	28	42	31	47	35	26	39	29	44	33	48
262	0	23	35	26	38	29	44	33	24	36	27	41	30	45	34	25	38	28	42	32	47
255	1	23	34	25	37	28	42	32	24	35	26	40	30	44	33	25	37	28	41	31	46



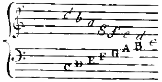
Tab I

Pitch. ¼ tones. The numbers of beats in 15 Seconds, of the Vths in the System of mean tones.

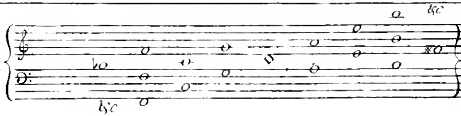
Pitch	¼ tones																				
293	4	23	35	26	39	29	44	33	24	37	27	41	30	46	34	26	38	29	43	32	48
285	3	23	34	25	38	28	43	32	24	36	27	40	30	45	33	25	37	28	42	31	47
277	2	22	33	25	37	28	41	31	23	35	26	39	29	43	32	24	36	27	40	30	45
269	1	22	32	24	36	27	40	30	23	34	25	38	28	42	31	24	35	26	39	29	44
262	0	21	31	23	35	26	39	29	22	33	24	37	27	41	31	23	34	26	38	29	43
255	1	20	30	23	34	25	38	28	21	32	24	36	27	40	30	22	33	25	37	28	42



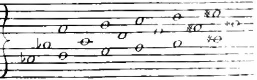
Tab. IV.

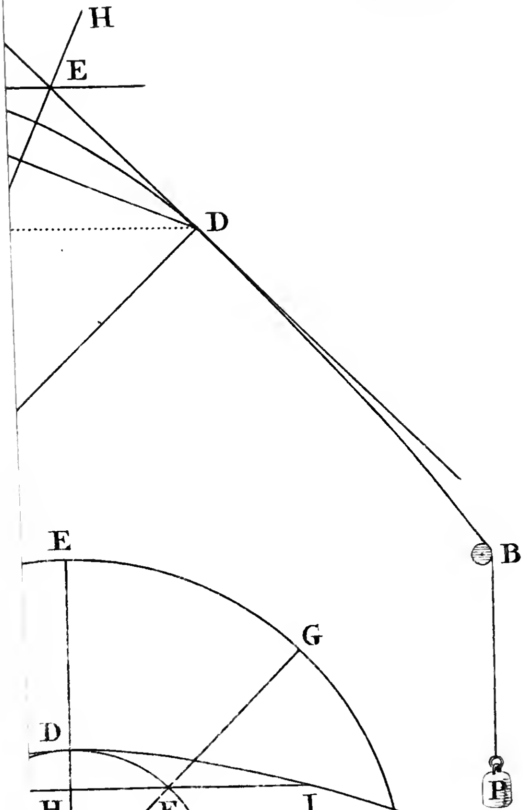


Tab. III



Tab. v.





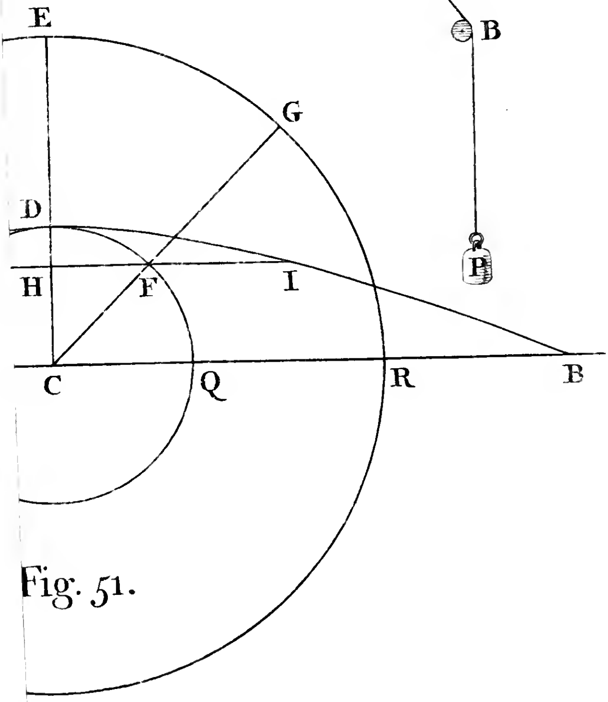


Fig. 51.

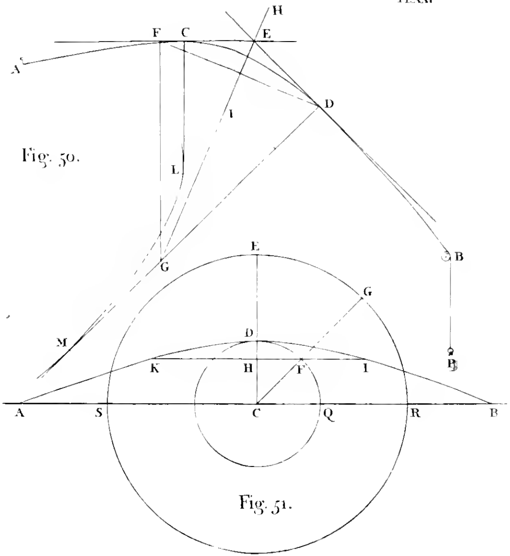
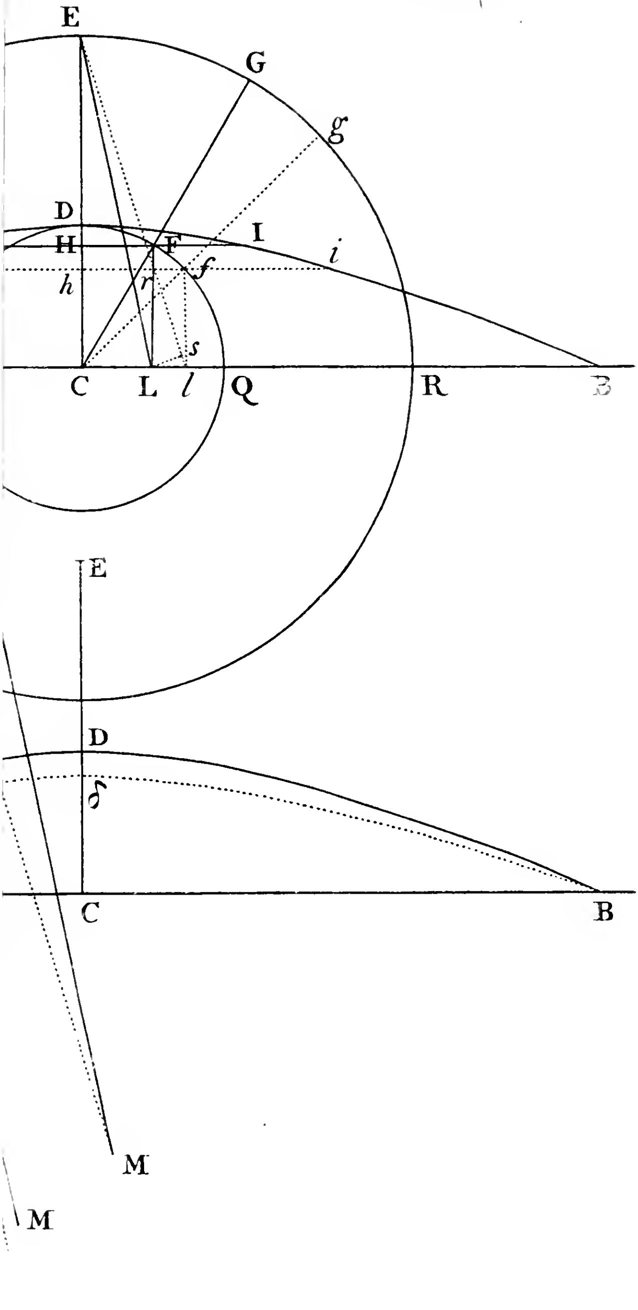


Fig. 50.

Fig. 51.

Pl. XXII.



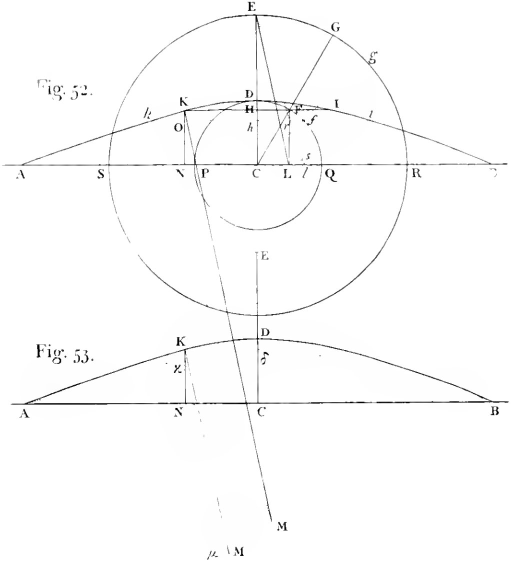


Fig. 52.

Fig. 53.

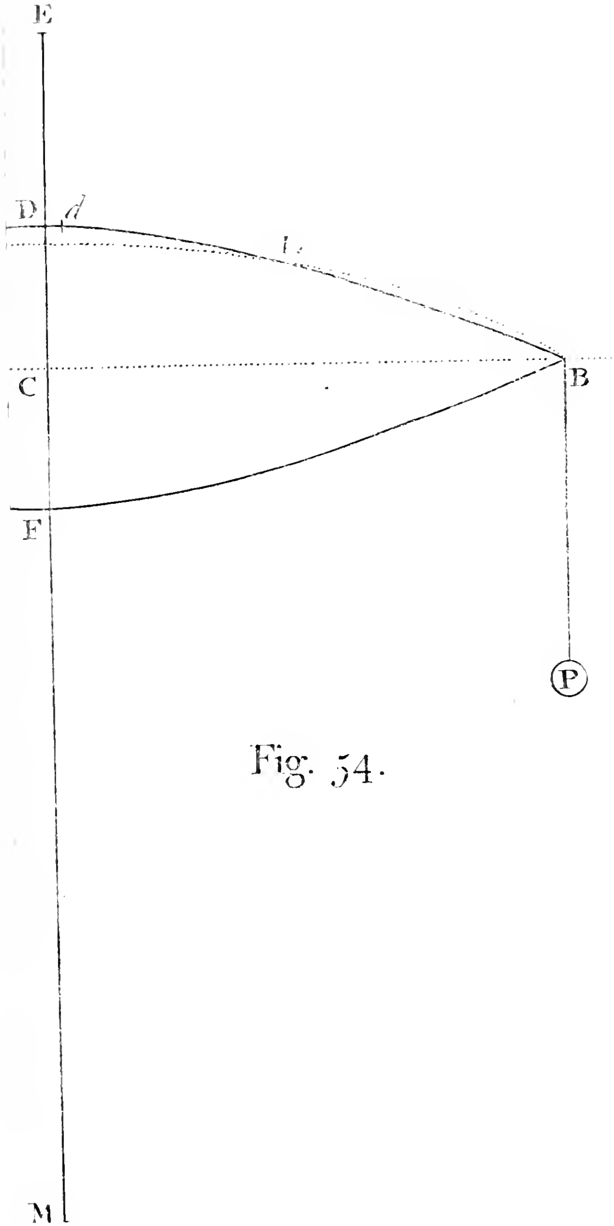


Fig. 54.

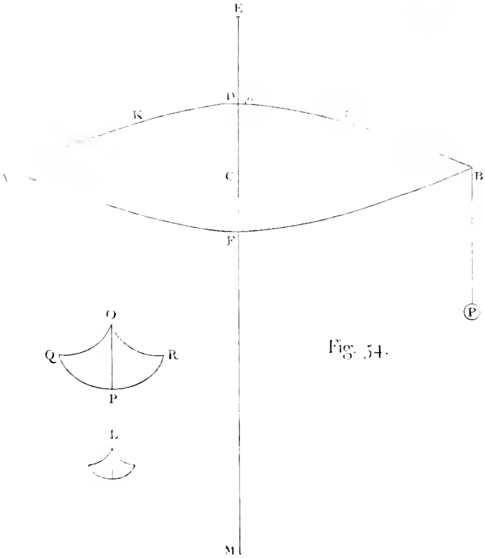


Fig. 54.

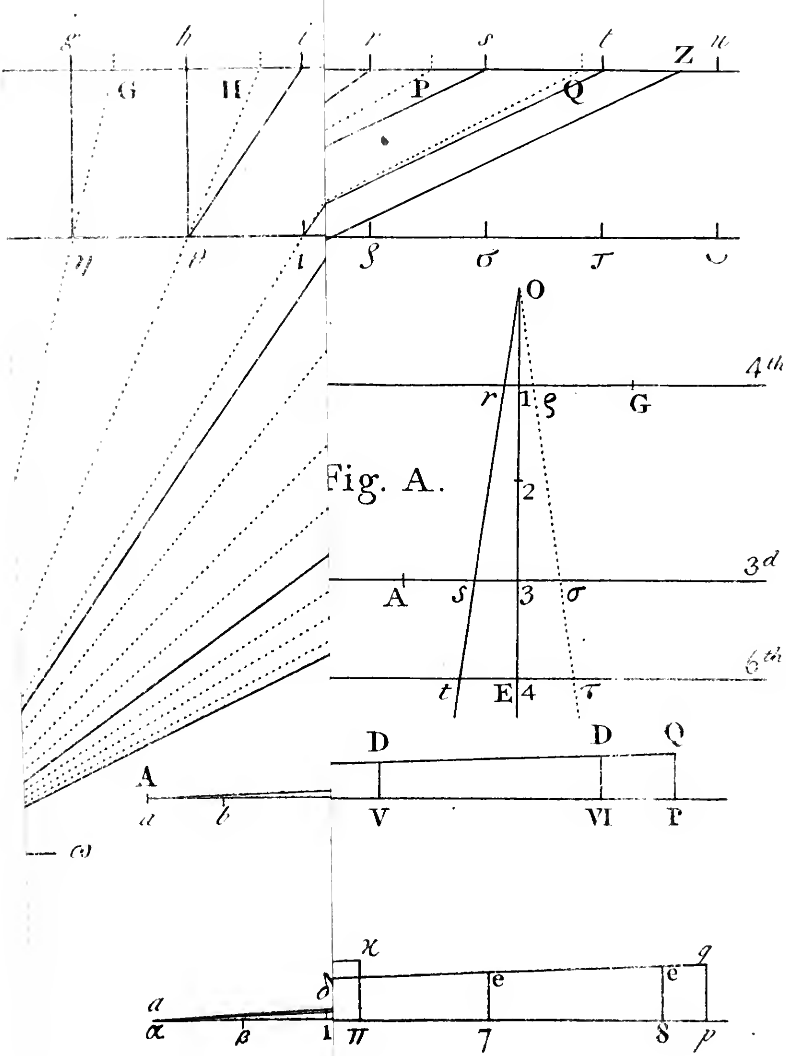

g
h
i
r
s
t
u
Z
G
H
P
Q
η
θ
ι
ϱ
σ
τ
υ
O
4th
r
1
ϱ
G
2
3d
A
s
3
σ
6th
t
E
4
τ
D
D
Q
A
a
b
V
VI
P
ω
κ
q
e
e
a
δ
α
β
1
π
7
8
p

Fig. A.

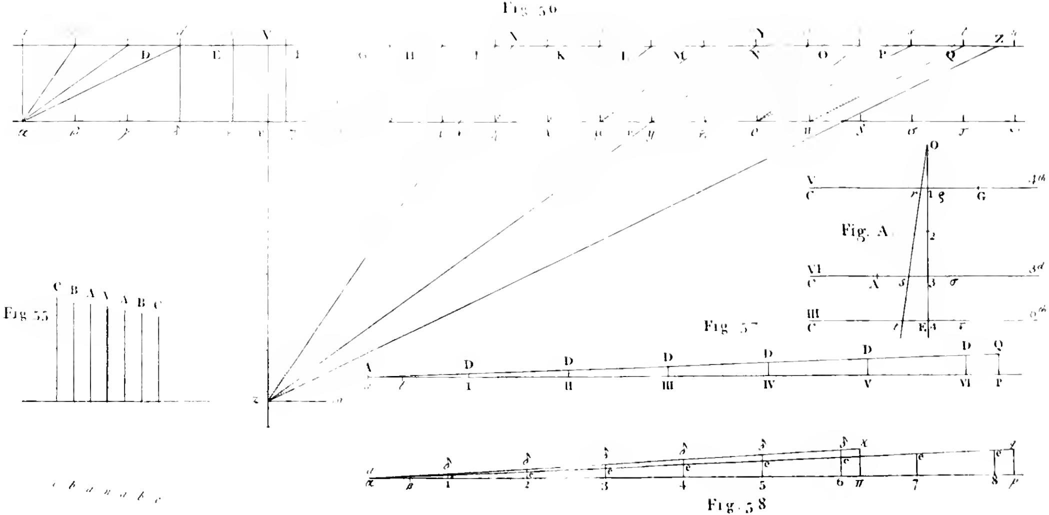


Fig. 56

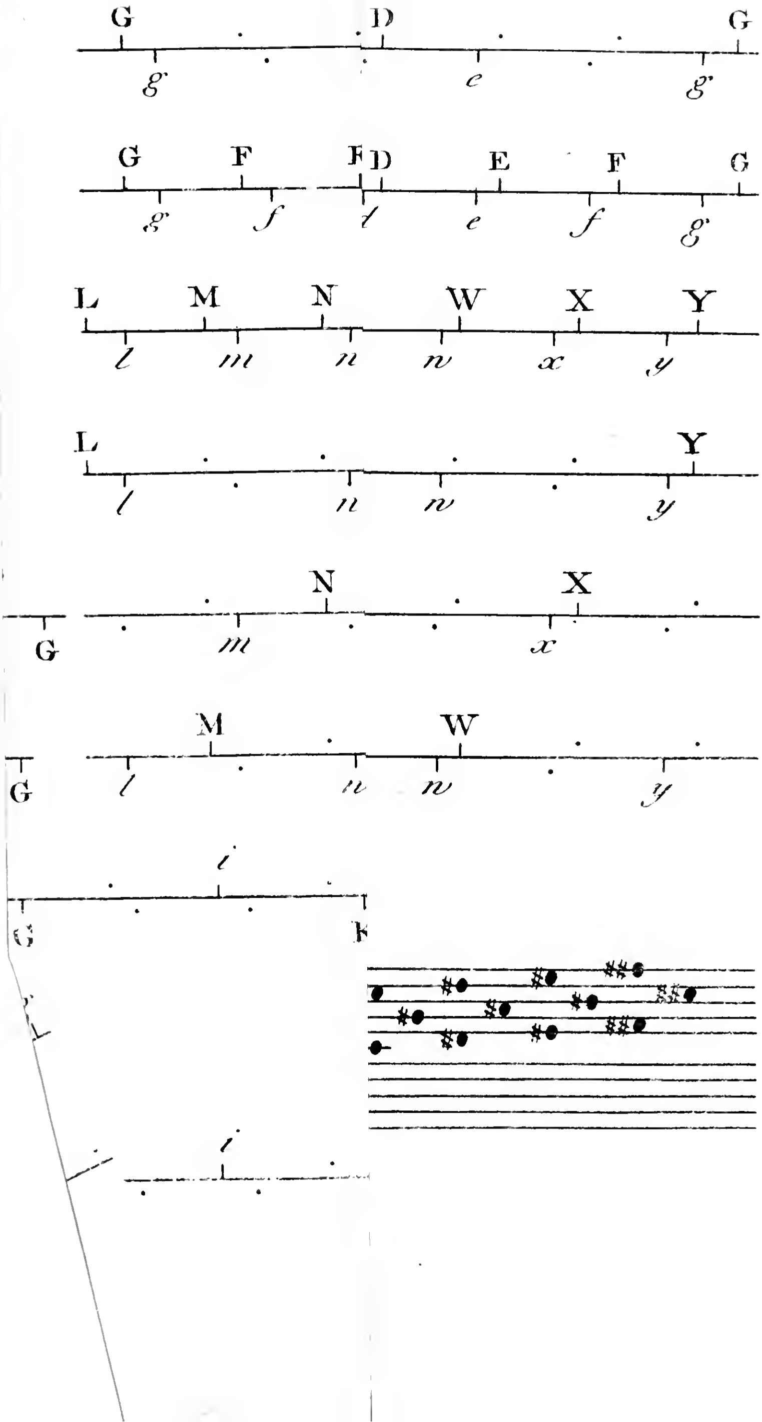
Fig. 55

Fig. A

Fig. 57

Fig. 58

Pl.XXV.


G D G
g e g
G F F D E F G
g f d e f g
L M N W X Y
l m n w x y
L Y
l n w y
N X
G m x
M W
G l n w y
i
G K
i

Pl XXV

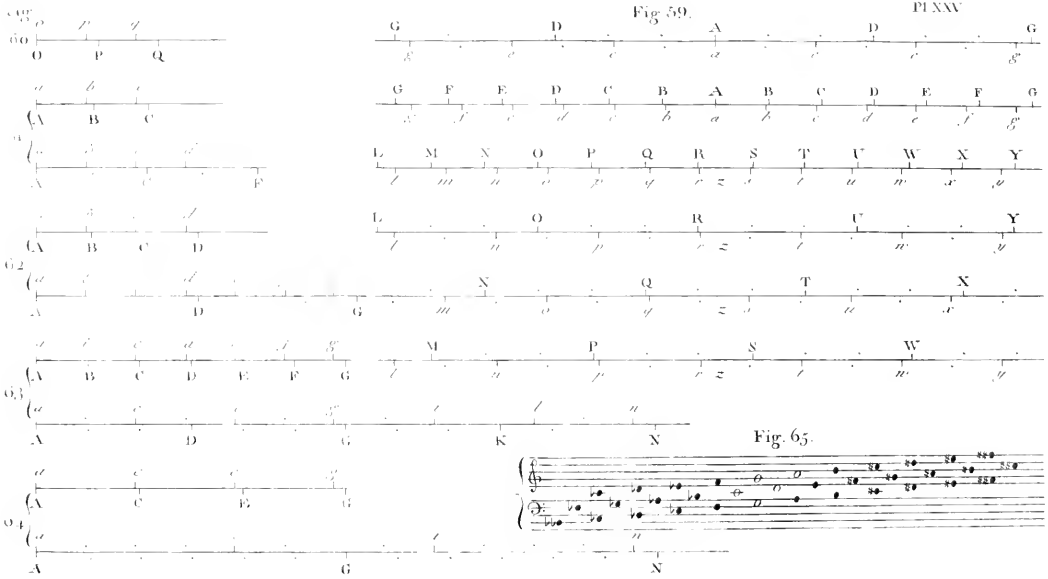


Fig. 59.

Fig. 65.

Fig. 67.

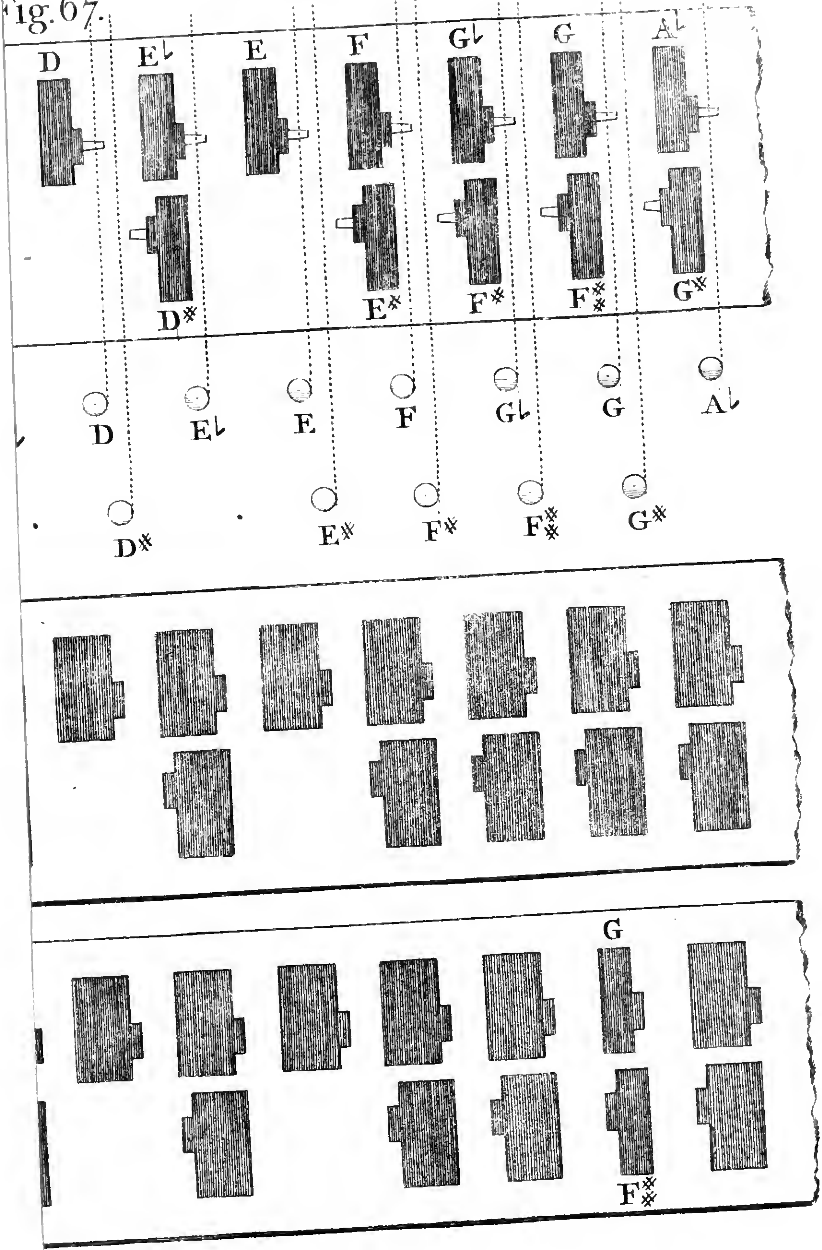


Fig.66.

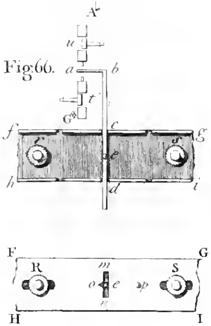
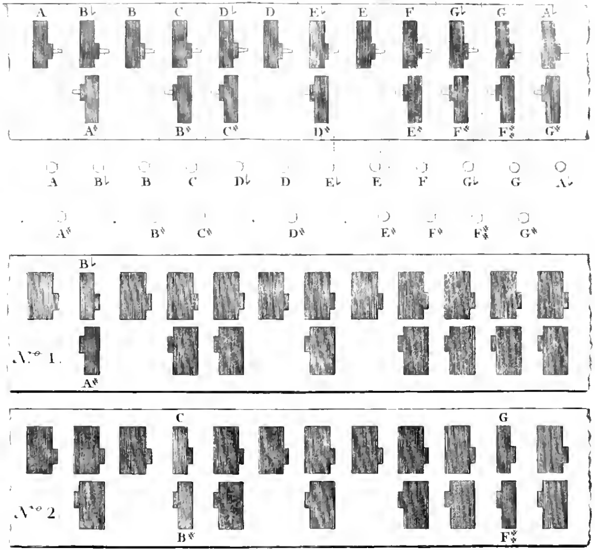
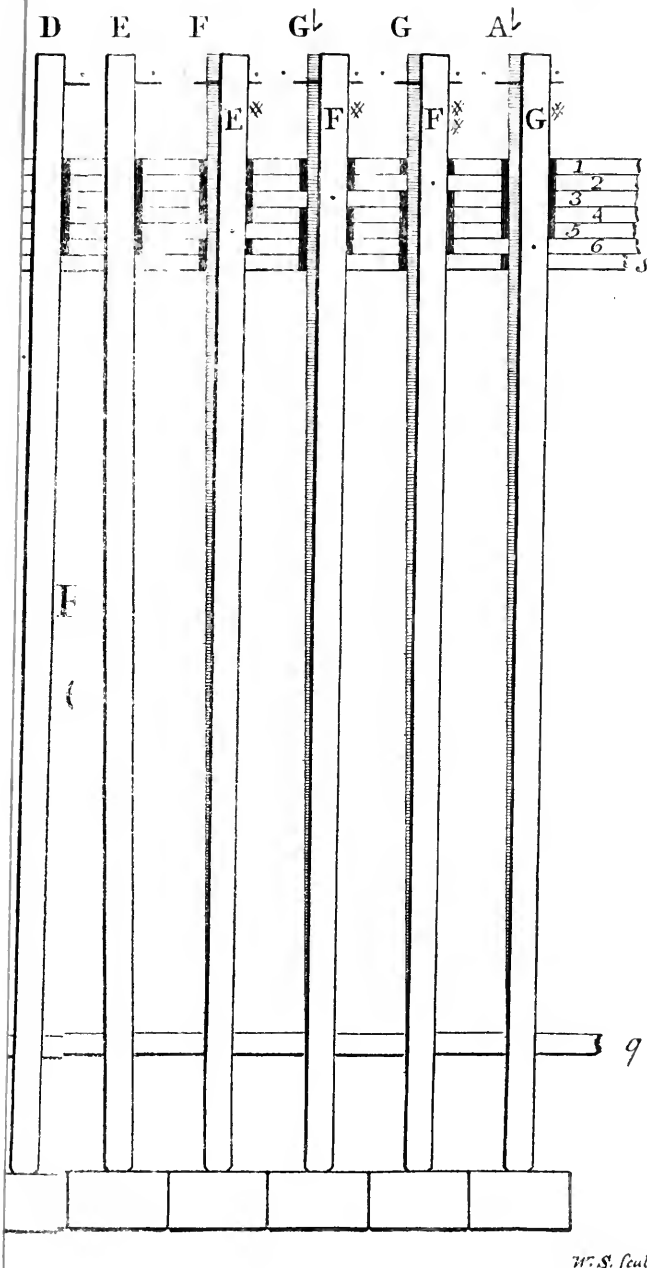

A♭
u
a
b
t
G♯
c
f
g
e
h
d
i
F
G
R
m
S
o
e
p
n
H
I

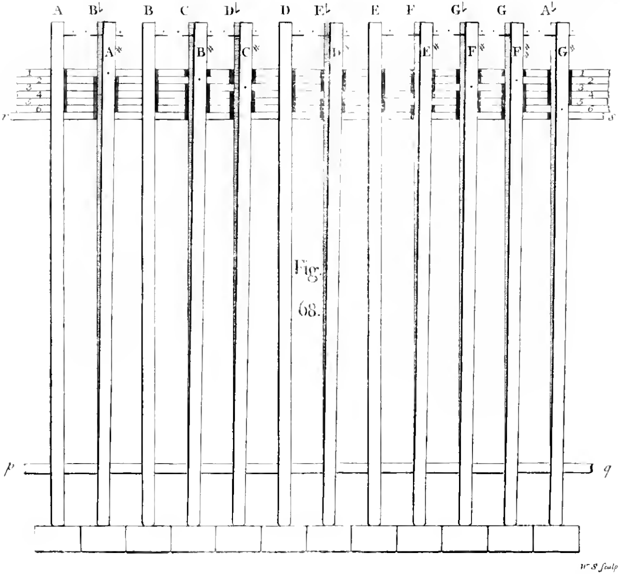
Fig 67.


A B♭ B C D♭ D E♭ E F G♭ G A♭
A♯ B♯ C♯ D♯ E♯ F♯ F𝄪 G♯
A B♭ B C D♭ D E♭ E F G♭ G A♭
A♯ B♯ C♯ D♯ E♯ F♯ F𝄪 G♯
B♭
N.° 1.
A♯
C
G
N.° 2.
B♯
F𝄪

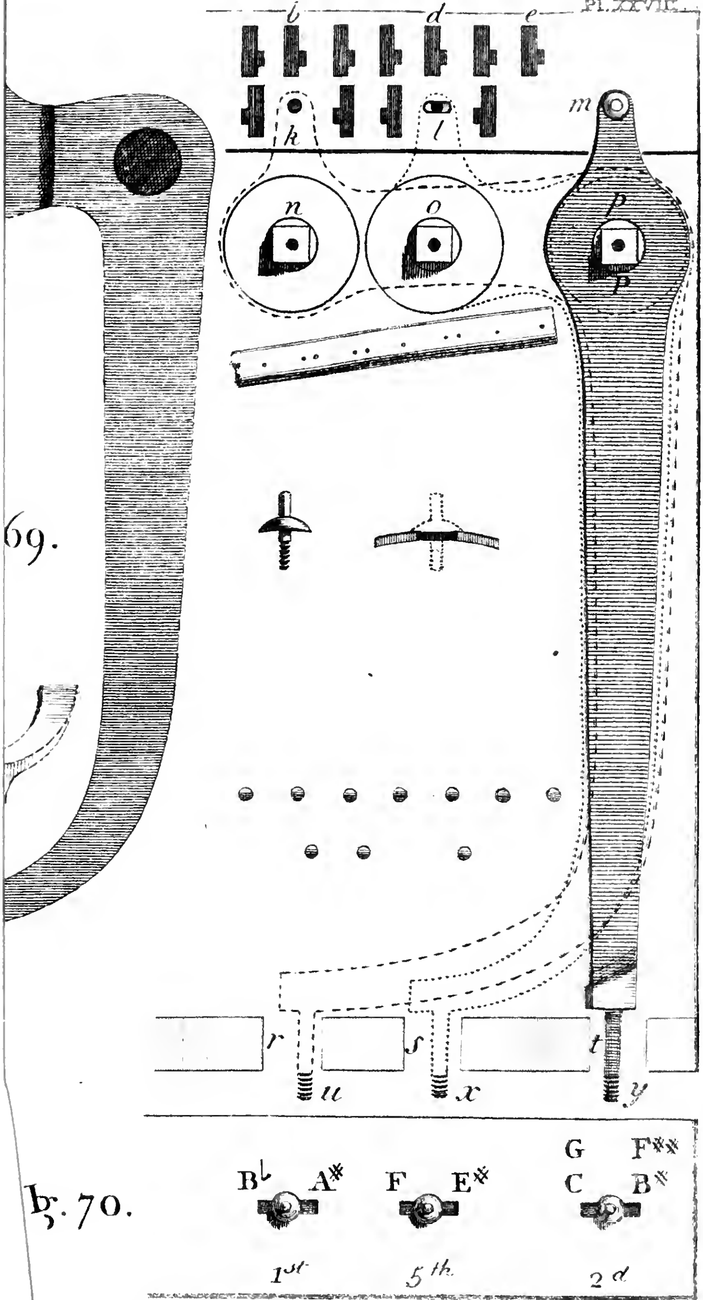
Pl.XXVII.



W. S. sculp.


A
B♭
B
C
D♭
D
E♭
E
F
G♭
G
A♭
A♯
B♯
C♯
D♯
E♯
F♯
F𝄪
G♯
r
s
p
q
Fig. 68.
Wᵐ S. sculp

Pl. XXVIII.



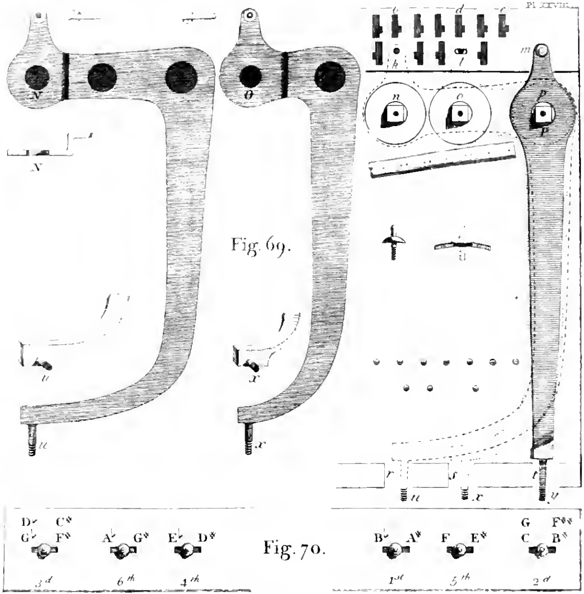

Pl. XXVIII
N
O
n
o
p
m
k
l
b
d
c
u
x
y
r
s
t
D♭ C♯
G♭ F♯
3d
A♭ G♯
6th
E♭ D♯
4th
B♭ A♯
1st
F E♯
5th
G F♯♯
C B♯
2d

Fig. 69.

Fig. 70.